HOW SHOULD MONETARY POLICY RESPOND TO ASSET-PRICE BUBBLES?

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Abstract

We present a simple model of the macroeconomy that includes a role for an asset-price bubble, and derive optimal monetary policy settings for two policy-makers. The first policy-maker, a sceptic, does not attempt to forecast the future possible paths for the asset-price bubble when setting policy. The second policy-maker, an activist, takes into account the complete stochastic implications of the bubble when setting policy.

We examine the optimal policy recommendations of these two policy-makers across a range of plausible assumptions about the bubble. We show that the optimal monetary policy recommendations of the activist depend on the detailed stochastic properties of the bubble. There are some circumstances in which the activist clearly recommends tighter policy than that of the sceptic, while in other cases, the appropriate recommendation is to be looser than the sceptic. Other things equal, the case for 'leaning against' a bubble with monetary policy is stronger the lower the probability of the bubble bursting of its own accord, the larger the efficiency losses associated with big bubbles, and the higher the assumed impact of monetary policy on the bubble process.

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1. Introduction

Asset-price bubbles pose difficult problems for monetary policy, and despite considerable debate no consensus has yet emerged on the appropriate strategy for monetary policy-makers in the presence of such bubbles.

Different views about the appropriate role of monetary policy in the presence of asset-price bubbles do not arise primarily because of differences about the objectives of monetary policy. These objectives, it is usually agreed, are to maintain low inflation and to limit the volatility of inflation and output, thereby contributing to stability in both the macroeconomy and the financial system. Rather, the different views are about how best to achieve these objectives.

One view is that monetary policy should do no more than follow the standard precepts of inflation targeting. Proponents of this view would acknowledge that rising asset prices often have expansionary effects on the economy, and might sometimes also provide a signal for incipient inflationary pressures, so that some tightening of monetary policy might be appropriate. According to this view, however, policy should only respond to observed changes in asset prices to the extent that they signal current or future changes to inflation or the output gap. There should be no attempt to use policy either to gently lean against a suspected asset-price bubble while it is growing or, more aggressively, to try to burst it. This view of the appropriate monetary policy response to asset-price bubbles has been put recently by Bernanke (2002).

An alternative view is that monetary policy should aim to do more than respond to actual and expected developments in inflation and the output gap. Cecchetti, Genberg and Wadhwani (2003), prominent proponents of this alternative view, put the argument in these terms:

> ... central banks seeking to smooth output and inflation fluctuations can improve ... macroeconomic outcomes by setting interest rates with an eye toward asset prices in general, and misalignments in particular ... *Raising interest rates modestly as asset prices rise above what are estimated to be warranted levels, and lowering interest rates modestly when asset prices fall below warranted levels, will tend to offset the impact on output and inflation of [asset-price] bubbles, thereby enhancing overall macroeconomic stability*. In addition, if it were known that monetary policy would act to 'lean against the wind' in this way, it might reduce the probability of bubbles arising at all, which would also be a contribution to greater macroeconomic stability. (p 429, italics added)[1]

We argue here that it is not clear that central banks should follow this advice. There is no universally optimal response to bubbles, and the case for responding to a particular asset-price bubble depends on the specific characteristics of the bubble process.

We present a simple model of the macroeconomy that includes a role for an asset-price bubble, and derive optimal monetary policy settings for two policy-makers. The first policy-maker, a sceptic, makes no attempt to forecast future movements in asset prices when setting policy, perhaps because she does not believe in the existence of the bubble or, alternatively, does not believe that monetary policy should actively respond to it. Her policy settings define the standard inflation-targeting benchmark in our model. The second policy-maker, an activist, takes into account the complete stochastic implications of the bubble when setting policy.

Once the bubble has formed, it is assumed to either grow each year with some probability, or to collapse and disappear. Crucially, and realistically, monetary policy in the model affects the economy with a lag, so that policy set today has its initial impact on the economy next year, by which time the bubble will have either grown further or collapsed.

For an activist policy-maker, it follows that there are two countervailing influences on monetary policy in the presence of the bubble. On the one hand, policy

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[1] Cecchetti *et al* are careful to argue that monetary policy should not target asset prices. To quote them again, 'we are *not* advocating that asset prices should be *targets* for monetary policy, neither in the conventional sense that they belong in the objective function of the central bank, nor in the sense that they should be included in the inflation measure targeted by monetary authorities' (2003, p 429, italics in the original).

should be tighter than the standard inflation-targeting benchmark to counter the expansionary effects of future expected growth in the bubble and, in some formulations, to raise the probability that the bubble will burst. On the other hand, policy should be looser to prepare the economy for the possibility that the bubble may have burst by the time policy is having its impact on the economy.

Which of these two influences dominates? For intermediate and larger bubbles – which are of most importance to policy-makers – we argue that it depends on the characteristics of the bubble process. There are circumstances in which the activist should recommend tighter policy than the sceptic. This is likely to be the appropriate activist advice when one or more of the following conditions applies: the probability that the bubble will burst of its own accord over the next year is assessed to be small; the bubble's probability of bursting is quite interest sensitive; efficiency losses associated with the bubble rise strongly with the bubble's size; or, the bubble's demise is expected to occur gradually over an extended period, rather than in a sudden bust.

Alternatively, however, when these conditions do not apply, it is more likely that the activist should recommend looser policy than the sceptic. This result makes clear that there is no single optimal rule for responding to all bubbles, and also illustrates the quite high level of knowledge of the future stochastic properties of the bubble that is required to set appropriate activist policy.

2. Model

Our model is an extension of the Ball (1999) model for a closed economy. In the Ball model, the economy is described by two equations:

$$y_t = -\beta r_{t-1} + \lambda y_{t-1} \tag{1}$$

$$\pi_t = \pi_{t-1} + \alpha y_{t-1} \tag{2}$$

where y is the output gap, r is the difference between the real interest rate and its neutral level, π is the difference between consumer-price inflation and its targeted rate, and α, β, and λ are positive constants (with $\lambda < 1$ so that output gaps gradually return to zero).

The Ball model has the advantage of simplicity and intuitive appeal. It makes the simplifying assumption that policy-makers control the real interest rate, rather than the nominal one. It assumes, realistically, that monetary policy affects real output, and hence the output gap, with a lag, and that the output gap affects inflation with a further lag. The values for the parameters α, β, and λ that Ball chooses for the model, and that we will also use here, imply that each period in the model is a year in length.[2]

We augment the model with an asset-price bubble. We assume that in year 0, the economy is in equilibrium, with both output and inflation at their target values, $y_0 = \pi_0 = 0$, and that the bubble has zero size, $a_0 = 0$. In subsequent years, we assume that the bubble evolves as follows:

$$a_t = \begin{cases} a_{t-1} + \gamma_t & , \text{ with probability } 1 - p_t \\ 0 & , \text{ with probability } p_t \, . \end{cases} \tag{3}$$

Thus, in each year, the bubble either grows by an amount, $\gamma_t > 0$, or bursts and collapses back to zero. For ease of exposition, in the rest of this section we will assume that γ_t is constant, $\gamma_t = \gamma$, but we will allow for a range of alternative possibilities in the results we report in Section 3. We also assume that once the bubble has burst, it does not re-form. To allow for the effect of the bubble on the economy, we modify Ball's two-equation model to read:

$$y_t = -\beta r_{t-1} + \lambda y_{t-1} + \Delta a_t \tag{4}$$

$$\pi_t = \pi_{t-1} + \alpha y_{t-1} \, . \tag{5}$$

In each year that the bubble is growing it has an expansionary effect on the economy, increasing the level of output, and the output gap, by γ. The bubble is, however, assumed to have no direct effect on consumer price inflation, although there will be consequences for inflation to the extent that the bubble leads the economy to operate with excess demand as it expands, and with excess supply when it bursts.

When the bubble bursts, the effect on the economy is of course contractionary – if the bubble bursts in year t, the direct effect on output, and the output gap, in that year will be $\Delta a_t = -(t-1)\gamma$. Thus, the longer the bubble survives, the greater will be the contractionary effect on the economy when it bursts.

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2 Ball's parameter values are $\alpha = 0.4$, $\beta = 1$ and $\lambda = 0.8$. Ball also adds white-noise shocks to each of his equations, which we have suppressed for simplicity.

We will assume that the evolution of the economy can be described by this simple three-equation system (Equations (3), (4) and (5)). But we distinguish between two policy-makers: a sceptic who doesn't try to second-guess asset-price developments, and an activist who believes that she understands enough about asset-price bubbles to set policy actively in response to them.[3]

We assume that the policy-makers observe in each year whether the bubble has grown further, or collapsed, before setting the interest rate for that year. Given the nature of the lags in the model, this year's interest rate will have no impact on real activity until next year, and on inflation until the year after that.

We also assume that the two policy-makers have the same preferences, and that they care about the volatility of both inflation and output. Thus we assume that in each year t, policy-maker p (activist or sceptic) sets the real interest rate, r_t, to minimise the weighted sum of the expected future squared deviations of inflation and output from their target levels, or in symbols, sets r_t to minimise

$$L = \sum_{\tau=t+1}^{\infty} \left[E_t^p(y_\tau^2) + \mu E_t^p(\pi_\tau^2) \right] \tag{6}$$

where μ is the relative weight on the deviations of inflation and E_t^p is the year t expectation of policy-maker p. In the results we show in the paper, we set $\mu = 1$, so that policy-makers are assumed to care equally about deviations of inflation from target and output from potential.

In setting policy each year, the sceptical policy-maker ignores the future stochastic behaviour of the bubble. Since certainty equivalence holds in the model in this setting, Ball (1999) shows that, for the assumed parameter values, optimal policy takes the form

$$r_t = 1.1y_t + 0.8\pi_t \tag{7}$$

which is a more aggressive Taylor rule than the 'standard' Taylor rule introduced by Taylor (1993), $r_t = 0.5y_t + 0.5\pi_t$.

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3 To draw the distinction more precisely, both policy-makers understand how the output gap and inflation evolve over time, as summarised by Equations (4) and (5). The activist also understands, and responds optimally to, the stochastic behaviour of the bubble, as summarised by Equation (3). The sceptic, by contrast, responds to asset-bubble shocks, Δa_t, when they arrive, but assumes that the expected value of future shocks is zero.

As the bubble grows, the sceptical policy-maker raises the real interest rate to offset the bubble's expansionary effects on the economy. But she does so in an entirely reactive manner, ignoring any details about the bubble's future evolution. Once the bubble bursts, output falls precipitously and the sceptical policy-maker eases aggressively, again in line with the dictates of the optimal policy rule, Equation (7).[4]

We assume that the activist policy-maker learns about the bubble in year 0, and hence takes the full stochastic nature of the bubble into account when setting the policy rate, r_t, from year 0 onwards. Once the bubble bursts, however, there is no further uncertainty in the model, and the activist policy-maker simply follows the modified Taylor rule, Equation (7), just like the sceptical policy-maker.

3. Results

In this section, we present optimal policy recommendations through time, assuming that the bubble survives and grows. We focus on the growth phase of the bubble's life because it is of most policy interest, as it generates the most disagreement about which policy approach is preferable. Once the bubble bursts, by contrast, there is general agreement that it is appropriate to ease aggressively to offset the contractionary effects of the bust.[5]

Our main aim is to compare the optimal policy recommendations of the sceptic with those of an activist, over a range of plausible alternative assumptions about the stochastic nature of the bubble. To do so in a meaningful way, it is necessary that the two policy-makers face an economy in the same state in each year. Since the current state of the economy depends on previous policy settings (as well as on the evolution of the bubble) we will assume throughout that the policy settings that are actually implemented each year are those chosen by the sceptic.

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4 We implicitly assume that the zero lower bound on nominal interest rates is not breached when policy is eased after the bubble bursts, so that the real interest rate can be set as low as required by Equation (7).

5 For completeness, Appendix A shows optimal interest rate recommendations both before and after the bursting of the bubble.

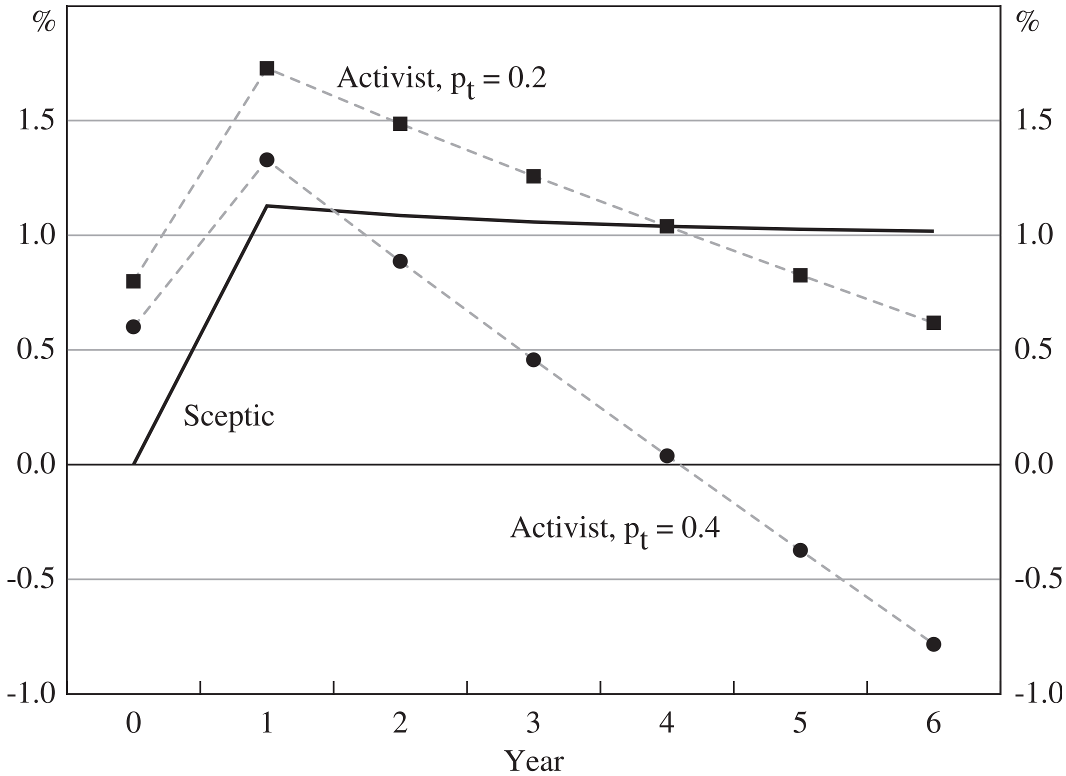
We can then meaningfully ask each year: given the state of the economy, what are the current optimal policy recommendations made by the different policy-makers? The activist's recommendations will depend on the assumptions she makes about the future possible paths of the bubble, while the sceptic's will not, since she assumes that future asset-price shocks have no expected effects.

3.1 Baseline Results: Policy Cannot Affect the Bubble

We begin with some simple baseline results. For these results, we assume that the bubble's direct expansionary effect on output in each year of its growth is a constant 1 per cent (i.e., $\gamma_t = 1$). Figure 1 shows the optimal policy choices made by the sceptic and two activists. We focus first on the sceptic, and then on the activists.

Figure 1: Real Interest Rate Recommendations While the Bubble Survives

Policy has no effect on the bubble



Notes: The sceptic implements policy in each year. Real interest rates are deviations from neutral.

Since the sceptic assumes that future asset-price shocks have no expected effects, she responds to the bubble only when its initial expansionary effects are manifest in year 1. As time proceeds and the bubble grows, she sets the policy interest rate in line with Equation (7), which is optimal given her beliefs about future asset-price shocks. Of course, were the bubble to burst, she would ease immediately (see Appendix A for further details).

An activist, deciding on optimal policy in year t, understands that if the bubble continues to grow, its direct effect on output next year will be +1 per cent, while if it bursts, the direct effect next year will be $-a_t$ per cent. If the probability of bursting each year is a constant, p^*, the bubble's expected direct effect on output next year is $(1-p^*)-p^*a_t$.

Certainty equivalence applies to this baseline version of the model.[6] It follows that the difference between the policy interest rates recommended by the activist, r_t^{ac}, and the sceptic, r_t^{sc}, depends only on their different assessments of the expected effect of the bubble on output next year. With the sceptic assuming that the bubble will have no expected effect on output next year, it follows that

$$r_t^{ac}-r_t^{sc}=(1-p^*)-p^*a_t\,. \tag{8}$$

Equation (8) implies that the activist will recommend tighter (easier) policy than the sceptic whenever, in probability-weighted terms, the expansionary effect on real activity from the bubble surviving is greater (less) than the contractionary effect from the bubble collapsing.

6 The model set-up is more complex than the standard set-up in which certainty equivalence applies. This is because, once the bubble bursts, there are no further asset-bubble shocks and hence, *ex ante*, the distribution of shocks is not independent through time. It is, therefore, not straightforward to demonstrate certainty equivalence. Nevertheless, Equation (8) in the text does follow and can be generalised to allow for alternative parameter values, time-varying bubble growth and/or probabilities of bubble collapse, provided that the evolution of the bubble remains independent of the actions of the policy-makers. The generalised equation is $r_t^{ac}-r_t^{sc}=\beta^{-1}[(1-p_{t+1})\gamma_{t+1}-p_{t+1}a_t]$ which, in particular, implies that $(r_t^{ac}-r_t^{sc})$ does not depend on α, λ or μ. A proof of this equation is provided in Appendix C.

For the results shown in Figure 1, we assume that the only difference between the two activists is that one assesses the probability that the bubble will burst each year as $p_t = p^* = 0.2$ (the 'durable-bubble activist'), while the other assesses it as $p_t = p^* = 0.4$ (the 'transient-bubble activist').[7]

In terms of their optimal policy recommendations, the two activists agree that policy should be tighter than the settings chosen by the sceptic for the first couple of years of the bubble's growth (including year 0, since that is when they learn about the bubble). Although they disagree about the details, they share the assessment that the continued probable growth of the bubble is a more important consideration for policy than the bubble's possible collapse.

The activists both understand, however, that as time proceeds, the bubble is getting bigger and the size of the prospective bust is also getting bigger. As a consequence, if the bubble survives for more than a year or two, the two activists no longer agree about whether policy should be tighter or looser than the modified Taylor-rule settings chosen by the sceptic. The durable-bubble activist recommends tighter policy because she assesses the probability of the bubble bursting to be small, but the transient-bubble activist recommends looser policy because her assessment is that this probability is larger.

If the bubble survives for long enough the two activists will again concur at least in the direction of their policy advice – they will both recommend looser policy than the sceptic because the possibility of the by-now-bigger bubble collapsing eventually dominates for them both.

In this case, then, the policy recommendations of an activist – and even whether she recommends tighter or looser policy than the benchmark settings chosen by the sceptic – depend crucially on her assessment of the probability that the bubble will collapse of its own accord. This is an important example of the general point that the activist's policy advice will depend critically on the detailed assumptions

7 Assuming $p_t = 0.2$ implies an average remaining life for the bubble of five years, while $p_t = 0.4$ implies an average remaining life of two and a half years.

she makes about the stochastic properties of the bubble. This is the central insight of the paper. We now show the relevance of this insight across a wide range of alternative assumptions about the bubble's stochastic behaviour.

3.2 Sensitivity Analysis[8]

3.2.1 Policy affects the probability that the bubble will burst

An obvious extension to the model is to assume that by setting tighter policy *this year*, the policy-maker can raise the probability that the bubble will burst *next year*. For simplicity, we initially assume a linear relationship between the interest rate and the probability of the bubble bursting:

$$p_t = p^* + \delta(r_{t-1} - r^*_{t-1})\,. \tag{9}$$

We assume that $\delta = 0.1$, so that a 1 percentage point rise in the real interest rate this year raises the probability of the bubble bursting next year by 0.1, subject to the constraint that $0 \leq p_t \leq 1$. The path of interest rates, r^*_t, $t \geq 0$, is the optimal path chosen by the sceptical policy-maker.[9]

As before, we assume that the bubble's direct expansionary effect on output in each year of its growth is a constant 1 per cent (i.e., $\gamma_t = 1$). Figure 2 shows the optimal policy recommendations made by the sceptic and two activists. The two activists again differ only in their assessment of the bubble's probability of collapse. Both believe that this probability is given by Equation (9), but the

[8] Most of the extensions we examine in this section imply that certainty equivalence no longer applies to the model (the exceptions are the bubble that collapses over two or more years and the rational bubble), in which case the results must be derived by numerical optimisation. To simplify the numerical problems, we assume that if the bubble survives until year 14 (which is a very unlikely event for all the parameter values we consider) then it bursts with certainty in that year. For earlier years, this assumption is only relevant for the policy choices of the activist policy-maker.

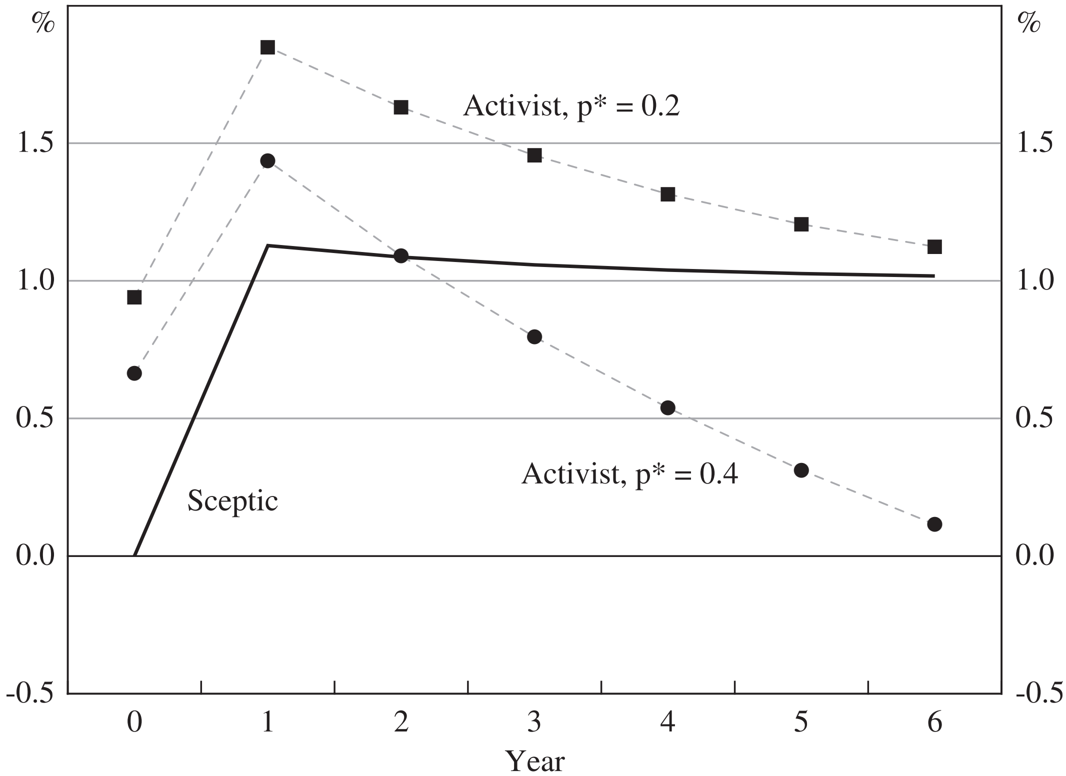
[9] We choose the functional form in Equation (9) so that, for the benchmark policy settings chosen by the sceptic, $p_t = p^*$ for all t. The results generated using an alternative functional form, $p_t = p^* + \delta r_{t-1}$, are qualitatively very similar to those shown.

durable-bubble activist believes that $p^* = 0.2$, while the transient-bubble activist believes that $p^* = 0.4$.

The sceptic's optimal policy profile is the same as in Figure 1, because she ignores the future stochastic details of the bubble. By contrast, it is optimal for the activists to recommend tighter policy than they would recommend if they had no influence on the bubble, as can be seen by comparing the activist profiles in Figures 1 and 2. By tightening somewhat, the activists reduce the probability that the bubble will grow further and be more disruptive to the economy when it ultimately bursts. Nevertheless, the optimal policy continues to depend, sensitively, on the activist's assessment of the bubble's probability of collapse, just as it did when the activists could not affect the bubble.

Figure 2: Real Interest Rate Recommendations While the Bubble Survives

Policy affects the bubble's probability of bursting



Notes: The probability of the bubble bursting is given by Equation (9) with $\delta = 0.1$. The sceptic implements policy in each year. Real interest rates are deviations from neutral.

It is also of interest to see how the results change when we vary the sensitivity to interest rates of the bubble's probability of collapse. For this exercise, we assume a monotonically increasing, but non-linear, relationship between interest rates and this probability, to avoid a corner-solution problem with the linear form (explained shortly). The relationship we assume is

$$p_t = \frac{1}{1+e^{a(r_{t-1}-r^*_{t-1})+b}} \tag{10}$$

where $a = -\delta/(p^*(1-p^*))$ and $b = ln((1-p^*)/p^*)$. For this functional form, $p_t = p^*$ when $r_{t-1} = r^*_{t-1}$ and $\partial p_t/\partial(r_{t-1} - r^*_{t-1}) = \delta$ when this derivative is evaluated at $r_{t-1} = r^*_{t-1}$. These two features are also features of the linear form, Equation (9). The advantage of the non-linear form, Equation (10), is that, while raising last year's interest rate, r_{t-1}, raises the probability that the bubble will burst this year, p_t, it cannot drive that probability to one, as can occur with the linear form.[10]

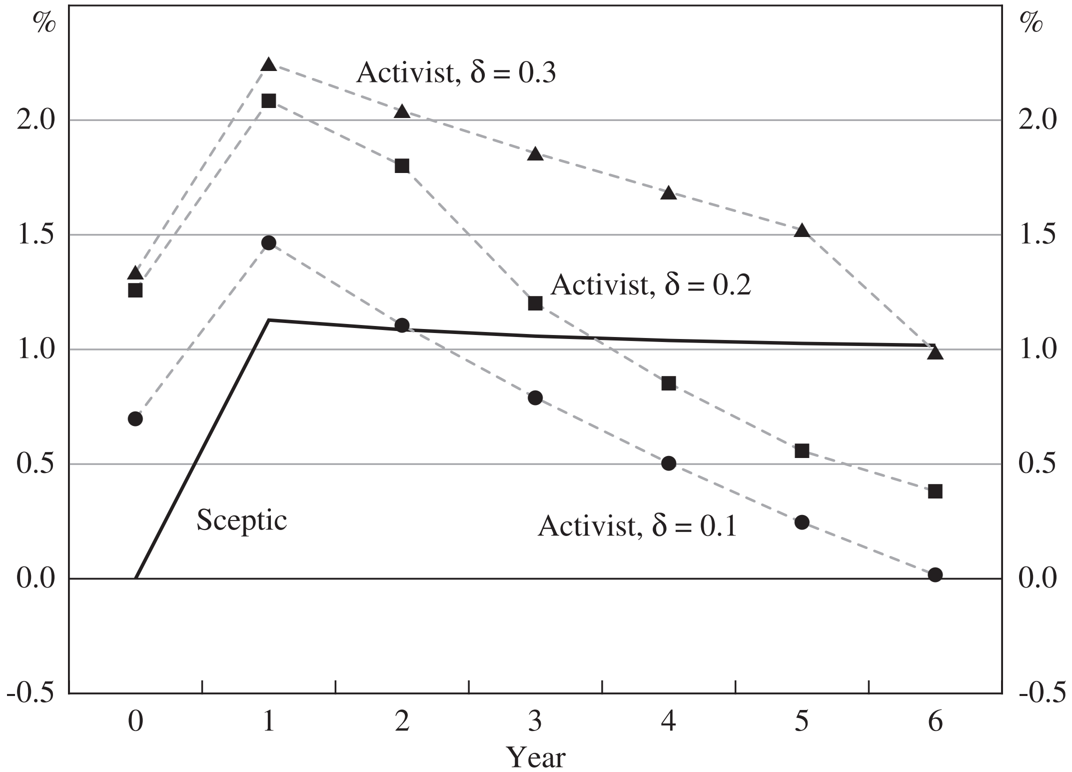
Figure 3 shows a comparison of optimal interest rate recommendations for the sceptic and three activists. The activists assume that the bubble's probability of bursting is given by Equation (10) with $p^* = 0.4$ (except $p_{14} = 1$), but they assume three different degrees of interest-rate sensitivity: $\delta = 0.1$, $\delta = 0.2$ or $\delta = 0.3$.

The pattern of optimal interest rate recommendations is somewhat similar to those in Figures 1 and 2. When the bubble is very small, the activists all agree that policy should be tighter than the setting chosen by the sceptic. But this consensus among the activists evaporates as the bubble gets bigger, and from year 2 onward, first one and then two of the three activists recommend looser policy than the sceptic, while the activist who believes that the bubble is highly interest sensitive ($\delta = 0.3$) continues to recommend tighter policy, at least until year 6.

[10] It seems implausible that moderate rises in the real interest rate would burst the bubble with certainty; yet that is an implication of the linear form, Equation (9). Simulations of the linear model with $\delta > 0.1$ do indeed generate this outcome (results not shown). It is for this reason that we use the non-linear form for simulations with $\delta > 0.1$. As argued by Stockton (2003), one could also imagine that the relationship between the bubble's probability of collapse and the policy interest rate might be non-monotonic, with small interest rate rises *lowering* the subsequent probability of collapse. This would undoubtedly further complicate the optimal policy recommendations of an activist.

Figure 3: Real Interest Rate Recommendations While the Bubble Survives

Varying the interest sensitivity of the probability of bursting



Notes: The probability of the bubble bursting is given by Equation (10) with $p^* = 0.4$. The sceptic implements policy in each year. Real interest rates are deviations from neutral.

3.2.2 Allowing for efficiency losses

A second natural extension is to allow for efficiency losses associated with the bubble. There are two broad ways to motivate the idea of efficiency losses. They can be motivated in terms of the economically inefficient physical over-investment that is put in place in response to asset-price rises that are not based on fundamentals, or in terms of the damage done to the financial system when the bubble bursts.

Either way, it seems plausible that the efficiency losses rise with the size of the bubble. To account for these losses, we re-formulate the policy problem as setting r_t to minimise

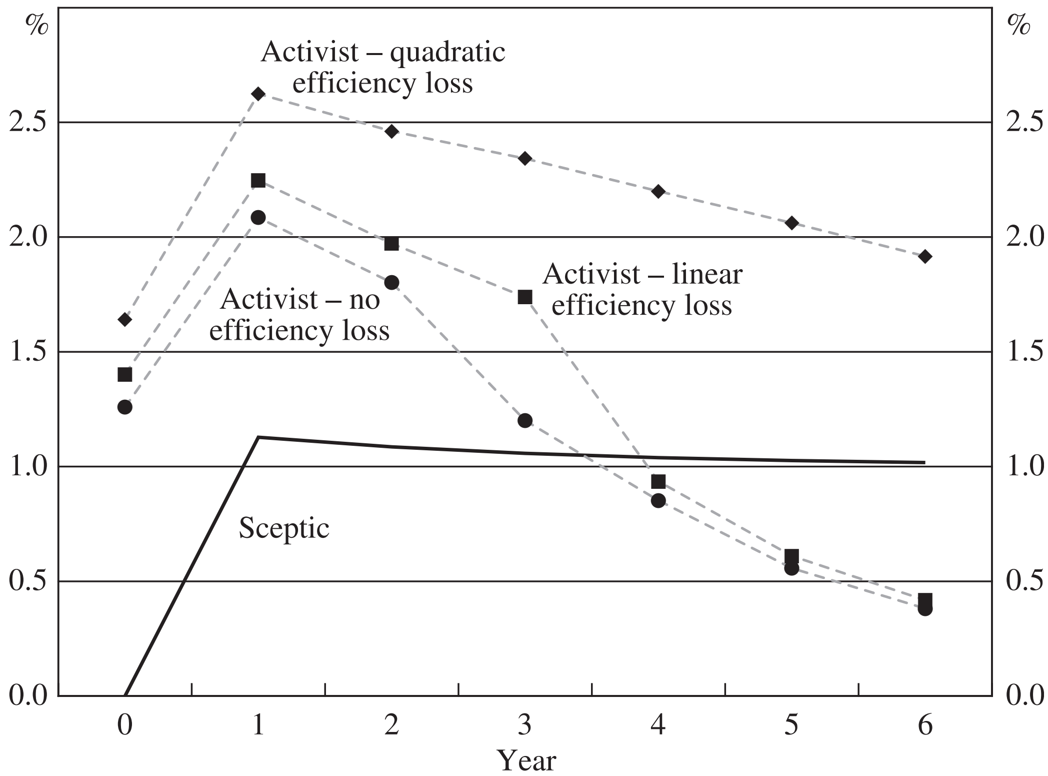
$$L = E_t^p\left[\max(a_\tau)\right]^\kappa + \sum_{\tau=t+1}^{\infty}\left[E_t^p(y_\tau^2) + E_t^p(\pi_\tau^2)\right] \tag{11}$$

where we assume that the efficiency losses rise either linearly with the maximum size of the bubble ($\kappa = 1$) or with the square of this maximum size (quadratic case, $\kappa = 2$). We also assume, as before, that the relative weight on inflation deviations, μ, takes a value of one. Since the sceptic ignores the bubble, we assume for her that $E_t^{sc}[\max(a_\tau)]^\kappa = 0$.

Figure 4 shows a comparison of optimal interest rate settings for the sceptic and three activists. The activists all assume that the bubble's probability of bursting is given by Equation (10) with $p^* = 0.4$, and with interest-rate sensitivity, $\delta = 0.2$. The first activist, however, makes no allowance for efficiency losses, and hence minimises the standard loss function, Equation (6). The second activist assumes linear efficiency losses, while the third assumes quadratic losses, and so they minimise the loss function, Equation (11), assuming appropriate values for κ.

Figure 4: Real Interest Rate Recommendations While the Bubble Survives

Allowing for efficiency losses associated with the bubble



Notes: The probability of the bubble bursting is given by Equation (10) with $p^* = 0.4$ and $\delta = 0.2$. The sceptic implements policy in each year. Real interest rates are deviations from neutral.

As previous figures have shown, being able to raise the probability of the bubble bursting gives an incentive to the activist policy-maker to tighten policy somewhat. Figure 4 shows that taking account of efficiency losses associated with an asset-price bubble raises this incentive further, and therefore further raises the optimal interest rate recommendations of the activist. Moreover, if efficiency losses associated with the bubble are assumed to rise sufficiently rapidly with the maximum size of the bubble, then the incentive for the activist to recommend tighter policy than the sceptic is a strong one.

3.2.3 Policy affects the bubble's growth

A further natural extension to the simple version of the model involves assuming that, rather than affecting the probability of the bubble bursting, the activist policy-maker can, by setting tighter policy this year, reduce the extent of the bubble's growth next year if it survives. For the simulations we show for this case, we assume that $p_t = p^* = 0.4$ (except $p_{14} = 1$) and that

$$\gamma_t = 1 - \phi(r_{t-1} - r^*_{t-1}) \,. \tag{12}$$

For reasons we discuss shortly, only large values of the parameter ϕ generate significantly changed behaviour by the activist policy-maker. We therefore assume that $\phi = 1$, so that by setting policy 1 percentage point higher than the sceptic this year, the bubble's growth next year is reduced from 1 per cent to nothing.[11] As above, the path of interest rates defined by r^*_t, $t \geq 0$, is the optimal path chosen by the sceptical policy-maker assuming $\gamma_t = 1$.

Figure 5 shows a comparison of optimal interest rate recommendations for the sceptic and two activists. Both activists assume that the bubble's growth is given by Equation (12), but one assumes no interest-rate sensitivity, $\phi = 0$, while the other assumes high sensitivity, $\phi = 1$.[12]

For every year apart from year 0, being able to reduce the bubble's growth induces the activist policy-maker to recommend tighter policy than she otherwise would.

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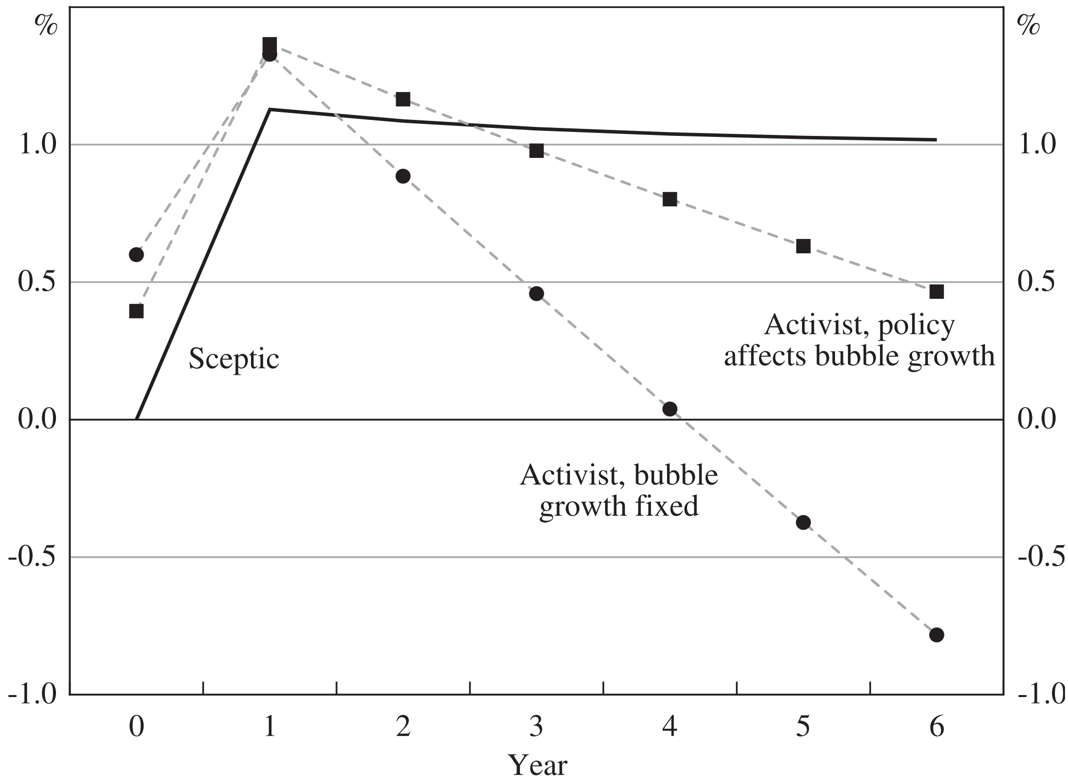
[11] If the bubble survives, it would again be necessary to set policy 1 percentage point higher than the sceptic to ensure that the bubble did not grow in the subsequent year. Given the effects of continually tight policy on the rest of the economy, it is perhaps not surprising that being able to raise the probability that the bubble will burst has more influence on optimal policy than simply being able to reduce its growth each year by setting tighter policy in each previous year.

[12] The results assuming no interest-rate sensitivity are equivalent to the baseline results shown in Figure 1 for the activist assuming $p_t = 0.4$.

Figure 5: Real Interest Rate Recommendations While the Bubble Survives

Policy affects the bubble's growth



Notes: The probability of the bubble bursting is $p_t = 0.4$. The sceptic implements policy in each year. Real interest rates are deviations from neutral.

The differences in the policy recommendations induced by this expectation are, however, less pronounced than the differences that arise when an activist policy-maker assesses the probability that the bubble will burst each year at $p_t = 0.2$ rather than $p_t = 0.4$, as can be seen by comparing Figures 1 and 5.

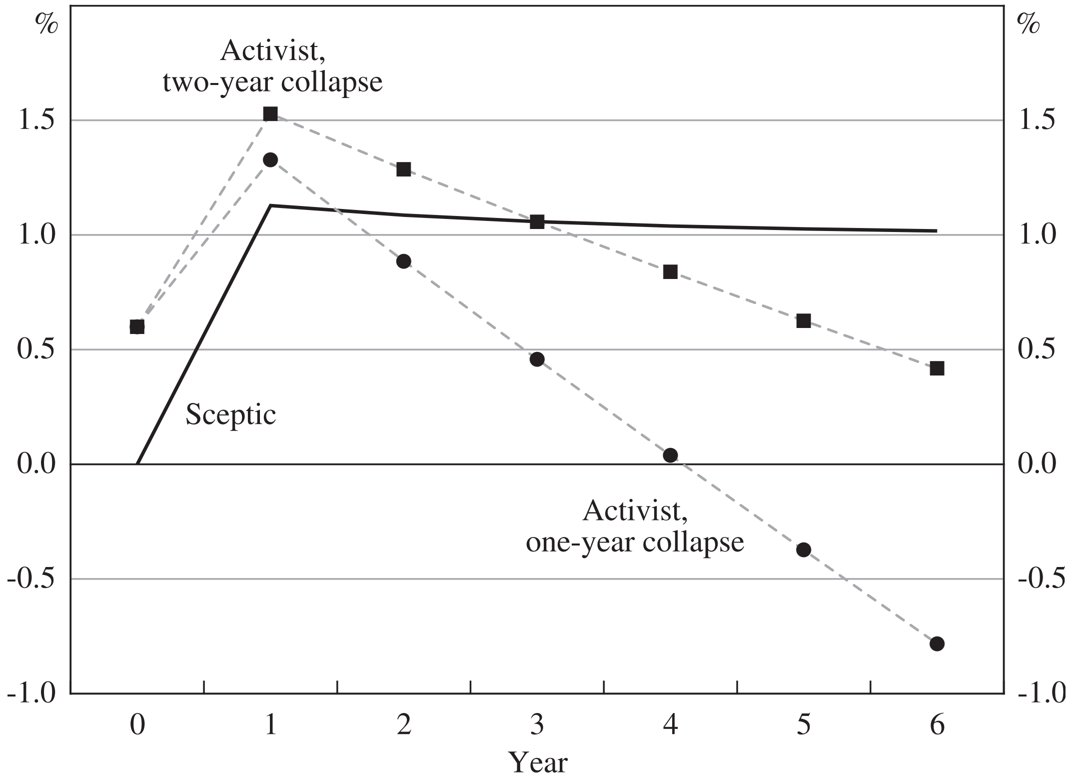
3.2.4 Bubbles that take two or more years to collapse

Another extension to the basic model involves assuming that, when the bubble collapses, it does so evenly over two or more years, rather than suddenly in one. In the examples we have examined until now, the activist must always confront the problem that, owing to the lag structure of the Ball model, policy can only respond to a collapsing bubble after the collapse is complete. This problem is reduced by assuming that the collapse occurs over two or more years rather than one.

Figure 6 shows results for the sceptic and two activists (one who assumes gradual, even, two-year collapse; the other, sudden), assuming that $p_t = p^* = 0.4$ (except $p_{14} = 1$) and that $\gamma_t = 1$. The activist who assumes that the bubble will collapse only gradually recommends tighter policy than the one who assumes that it will be sudden, because of their different assessments of the bubble's expected effect on next year's output.

Figure 6: Real Interest Rate Recommendations While the Bubble Survives

Bubble takes two years to collapse



Notes: The probability of the bubble bursting is $p_t = 0.4$. The sceptic implements policy in each year. Real interest rates are deviations from neutral.

Nevertheless, the overall pattern of policy recommendations remains similar to earlier cases. As the size of the bubble grows, the 'gradually-bursting' activist eventually recommends looser policy than the sceptic does, for reasons that are by now familiar.

In cases in which the bubble is expected to collapse evenly over three or more years, the activist would recommend tighter policy than the sceptic for longer, while the bubble is growing, a result that follows from a straightforward extension to Equation (8).

3.2.5 A rational bubble

In the baseline results presented at the beginning of the section, we assumed that the asset-price bubble grew at a uniform rate, $\gamma_t = 1$, and that the probability of the bubble's collapse was constant through time. This seems to us a simple and intuitively appealing baseline case.

In this case, however, there is no arbitrage condition ruling out unexploited profit opportunities in the assets whose price rises constitute the bubble. Our baseline case is therefore not a 'rational' bubble. We do not see this as a shortcoming – to our minds, there is much evidence that the asset-price bubbles we see in modern industrial economies are not rational in this sense (see, for example, Shiller (2000)). Nevertheless, it is of interest to derive results for the case of a rational bubble.

Such a bubble arises from the actions of a rational investor who buys the relevant assets up to the point at which expected profits are driven to zero.[13] If the probability of collapse is constant, p^*, and the capital gain to the investor in year $t+1$ if the bubble collapses is $-a_t$, then a rational risk-neutral investor will be indifferent to holding the asset when the expected growth of the bubble, if it survives, is $\Delta a_{t+1} = a_t p^* / (1 - p^*)$. This is a geometrically growing bubble, rather than the constant-growth bubble that constituted our baseline case.[14]

The arbitrage condition that defines this rational bubble implies that the bubble's expected growth over the next year, $E_t^{ac} \Delta a_{t+1}$, is zero. In this case, however, the activist and the sceptic are making identical assumptions about the bubble's expected effect on next year's output. It follows that the activist will always recommend the same policy interest rate as the sceptic for a rational bubble,

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[13] We assume that the assets yield an annual return equal to the real interest rate, so that the expected profit relative to holding 1-year government bonds is determined by the expected capital gain on the assets.

[14] Note that, if the probability of collapse is not constant, a rational bubble need not grow at a constant geometrical rate.

provided she believes that the stochastic properties of the bubble are not affected by the actions of policy-makers, so that certainty equivalence holds.[15]

4. Discussion and Conclusions

Table 1 provides a summary of the results. For each set of assumptions, it shows, as time proceeds and the bubble grows, whether the activist would recommend tighter (+), looser (−) or the same (=) policy settings as the sceptic.

Table 1: Activist's Policy Recommendations While the Bubble Survives
Tighter (+), looser (−), or the same as (=) the sceptic's recommendation

Scenario	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6
Policy can't affect bubble						
$p_t = 0.2$	+	+	+	=	−	−
$p_t = 0.4$	+	−	−	−	−	−
Policy affects probability of bursting						
$p^* = 0.2$, $\delta = 0.1$	+	+	+	+	+	+
$p^* = 0.4$, $\delta = 0.1$	+	+	−	−	−	−
$p^* = 0.4$, $\delta = 0.2$	+	+	+	−	−	−
$p^* = 0.4$, $\delta = 0.3$	+	+	+	+	+	−
Linear efficiency losses	+	+	+	−	−	−
Quadratic efficiency losses	+	+	+	+	+	+
Policy affects bubble growth	+	+	−	−	−	−
Bubble bursts over two periods	+	+	+	−	−	−
Rational bubble	=	=	=	=	=	=

[15] This result relies on a number of implicit, simplifying assumptions about the economy. In particular, it relies on the assumptions that the effect on the output gap of changes in asset prices is proportional to the size of those changes, and that rational investors and the activist policy-maker agree on the exact stochastic details of the bubble. Relaxing either of these assumptions could generate different policy recommendations by the activist. For example, for a geometrically growing bubble, it could account for an activist policy-maker assessing the bubble's growth rate to be faster (slower) than 'rational' – say, $\Delta a_{t+1} = \chi a_t p^*/(1-p^*)$, with $\chi > 1$ ($\chi < 1$) – in which case the activist's policy recommendations would always be tighter (looser) than the sceptic's, for as long as the bubble survived.

There are several broad lessons worth highlighting from this summary. When the asset-price bubble is small enough, the activist policy-maker always (except in the case of the rational bubble) recommends tighter policy than the sceptic who ignores the future possible paths of the bubble. However, this result is of limited practical relevance. Although we have assumed that activist policy-makers learn about the nature of the bubble at its inception, in reality there is likely to be much doubt in the early stages about whether rising asset prices constitute a bubble. Asset-price bubbles rarely arise out of thin air – instead, they usually occur when the evolving economic fundamentals are consistent with some rise in asset prices. While there will always be some doubt about whether rising asset prices constitute a bubble, these doubts would seem particularly acute when the suspected deviation of asset prices from fundamentals remains small and has been short-lived. For these reasons, there would seem to be no strong case for central banks to respond to small asset-price misalignments.[16]

As the bubble grows, however, there are two developments with potentially conflicting implications for appropriate activist policy. On the one hand, an activist policy-maker should become increasingly confident that the observed asset-price rises *do* constitute a bubble, which should strengthen the case for responding actively to them. On the other hand, as the bubble grows, the potential negative effects from its eventual bursting will increase. Whether this constitutes an argument for tighter or looser policy will depend on the nature of the bubble.

The case for tightening is to offset the expansionary effects of future expected growth of the bubble and, in some formulations, to reduce the bubble's growth or help to burst it. As we have seen, there are circumstances in which this case is particularly compelling, in particular when: the probability that the bubble will burst of its own accord over the next year is assessed to be small; the bubble's probability of bursting is quite interest sensitive; efficiency losses associated with the bubble rise strongly with the bubble's size; or, the bubble's demise is

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[16] Cecchetti *et al* (2003) also make this point when they say 'our proposal [to raise interest rates modestly as asset prices rise above what are estimated to be warranted levels] does not call for central banks to respond to small misalignments. We agree that these are difficult to detect and are unlikely to have very strong destabilizing effects in any case' (p 440).

expected to occur gradually over an extended period, rather than in a sudden bust. Conversely, the case for loosening is strongest when these conditions are reversed, since in those circumstances it becomes increasingly important to allow for the contractionary impact that arises when the bubble bursts.[17] The stochastic process driving the bubble is thus crucial to determining which of these considerations predominates.[18]

Ultimately, the appropriate policy strategy is a matter for judgement. Since the optimal policy response at any point depends on the stochastic properties of the bubble, our results highlight the information requirements inherent in an activist approach. Where sufficient information about the bubble process is not available to the policy-maker, a robust approach, something along the lines of the one used by our sceptic, may be the best that can be achieved. Given sufficient information about the bubble process, an activist approach may be feasible, but our results suggest that the appropriate response to bubbles is not uniform. In particular, it may be optimal to 'lean against' some bubbles but not others, and hence the formulation of an activist strategy requires judgments to be made about the process driving the bubble and its likely sensitivity to monetary policy.

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[17] In a passage immediately following the one quoted in the previous footnote, Cecchetti *et al* (2003) say '... there are clearly times when egregious misalignments exist. Recent examples include Japanese stock and land prices in 1989, and the NASDAQ in late 1999 and early 2000. While some portion of these high price levels may have been justifiable based on fundamentals, few people would deny that a significant component was due to asset market disturbances. Ultimately, in terms of reducing inflation and output volatility, *it is important that central bankers respond to these large relatively "obvious" misalignments*' (p 440, italics added). When misalignments are large and relatively obvious, however, our results suggest that it may be unclear whether the appropriate policy response should be to raise interest rates modestly or to lower them, unless the policy-maker is able to make use of specific knowledge about the stochastic process driving the bubble.

[18] It is also possible that the probability of the bubble bursting of its own accord over the next year might rise as the bubble gets larger. If so, the case for looser, rather than tighter, policy by the activist is further strengthened, a point also made by Kent and Lowe (1997). For most of our simulations, we have assumed $p^* = 0.4$, implying an average remaining life for the bubble of two and a half years, which may be a more plausible assumption for intermediate and larger bubbles than $p^* = 0.2$, which implies an average remaining life of five years.

Appendix A: Policy Settings for a Bubble that Bursts in the Fifth Year

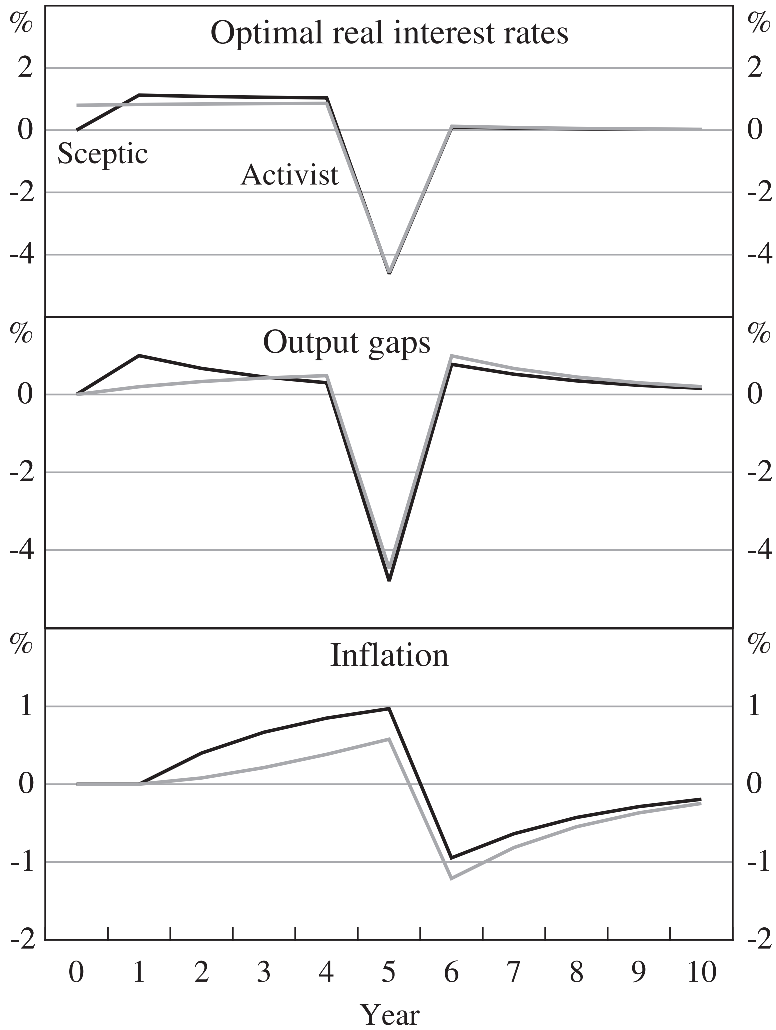
We assume a constant probability $p_t = 0.2$ that the bubble bursts in each year. In contrast to the simulations reported in the text, we allow both the sceptic and the activist to implement policy through time – so that the state of the economy depends on the identity of the policy-maker. Figure A1 shows results assuming that, as events turn out, the bubble grows for four years, during which time it has a direct expansionary effect on output of $\gamma = 1$ per cent in each year, and then bursts in the fifth year, with a direct contractionary effect on output of 4 per cent in that year.[19] The top panel shows the real interest rate profiles, r_t, set by the two policy-makers; the middle and bottom panels show the outcomes for the output gap, y_t, and the inflation rate, π_t.

While the bubble is growing, the paths for output and inflation generated by the sceptic's policy settings reflect the continued expansionary effects of the bubble. The activist responds more aggressively to these expansionary effects because she anticipates them, but nevertheless she does not offset them completely because of the possibility that the bubble may be about to burst. Therefore, even with the activist's optimal policy settings, output and inflation remain above target while the bubble survives.

The bursting of the bubble in year 5 generates a severe recession. Output falls by more than the direct contractionary effect of the bubble bursting, because policy in the previous year has been tighter than neutral to offset the bubble's expansionary effects. In response to the bubble's collapse, policy is eased aggressively. Despite using the same policy rule after the bubble bursts, the modified Taylor rule, Equation (7), the paths for the policy interest rate, output, and inflation are somewhat different for the two policy-makers because they have set different policy interest rates in earlier years.

[19] A bubble with a probability of bursting each year of $p_t = 0.2$ bursts on average in the fifth year.

Figure A1: Results for Bubble that Happens to Burst in the Fifth Year



Notes: Bubble's *ex ante* probability of bursting in each year is $p_t = 0.2$. Real interest rates are deviations from neutral; inflation rates are deviations from target.

Appendix B: Comparison with Kent and Lowe (1997)

Kent and Lowe (1997) present a simple model of an asset-price bubble that has similarities with ours. They derive optimal activist policy in their model for two of the cases we have examined: when the probability of the bubble collapsing is exogenous, and when this probability rises with the previous period's policy interest rate.[20]

Kent and Lowe show that, when policy cannot affect the bubble's probability of collapse, optimal activist policy generates average inflation in their period 2 equal to the central bank's target rate of inflation. When policy can affect the bubble's probability of collapse, however, optimal activist policy generates average inflation in period 2 *less than* the central bank's target rate of inflation (where the averages are calculated over all possible outcomes for the bubble).

The qualitative nature of these results carries over to our model set-up. When policy cannot affect the bubble, average inflation in every year of our model is also equal to the central bank's target. When policy can affect the bubble, however, either by affecting its probability of bursting or its rate of growth, average inflation from year 2 onward is always less than the central bank's target when activist policy is implemented.[21]

Kent and Lowe use their model to make the case that, when policy can affect the bubble's probability of collapse, it may make sense for the policy-maker to raise interest rates early in the life of the bubble, even though this will increase the likelihood of inflation being below target in the near term. As we have seen, this

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[20] Theirs is a 3-period model in which the bubble, which has formed in period 1, can either grow or collapse back to zero in period 2, and if it has grown, can grow further or collapse in period 3. Their periods should, therefore, probably be thought of as spanning more than one year.

[21] Recall that it takes two years for policy changes to affect inflation in our model. As for the Kent and Lowe model, in each year the averages must be calculated over all possible outcomes for the bubble, weighted by their appropriate probabilities. Calculated in this way, the averages are therefore equivalent to period-0 expectations.

general case – for tightening policy early in the life of the bubble – survives in our model. What our model adds to this story is that ‘early in the life of the bubble’ may not last very long. For many of our simulations, within a couple of years or so of the bubble’s inception, it is no longer clear whether optimal activist policy should be tighter or looser than the policy chosen by a sceptic.

Appendix C: Analytic Solution of the Optimal Policy Problem

In Section 3 we compared the recommendations of activist and sceptical policy-makers, when confronting various types of asset-price bubbles. For asset-price bubbles which are not influenced by policy, it is possible to derive an *analytic* formula for the difference between the recommendations of these two types of policy-makers. In this Appendix we outline this derivation in detail. The working is somewhat complicated owing to the contingent nature of the optimal policy problem facing an activist policy-maker in this setting.

The derivation proceeds in several stages. We first describe, in Section C.1, the contingent optimal policy problem facing an activist policy-maker, confronted by an asset-price bubble whose stochastic properties she cannot affect. Before attempting to treat this contingent problem, however, we then derive, in Section C.2, the analytic solution to the optimal policy problem facing an activist policy-maker in our Ball-style economy, when confronting a *known* set of future exogenous shocks to output.

Next, in Section C.3, we discuss how the optimal policy problem set out and solved in Section C.2 needs to be modified to handle the case of an exogenous asset-price bubble of the form treated in the main body of the paper – where an activist faces, not a known set of future exogenous shocks to output, but rather an *array* of possible different sets of future shocks, depending upon how the bubble develops in subsequent periods. This more general contingent optimal policy problem is then solved explicitly, in matrix terms, in Section C.4.

Finally, we then use this general solution to study the *difference* between the contingent optimal policy recommendations of activist and sceptical policy-makers confronting an asset-price bubble. As in Section 3, we assume throughout that policy is ultimately set in each period by the sceptic.[22]

[22] Note, however, that for the working which follows it does not in fact matter upon what basis policy is eventually set in each period. All that matters is that both activist and sceptical policy-makers face an economy in the *same* state in each year, when devising their recommendations on the optimal stance of monetary policy.

C.1 Nature of the Problem for an Activist Policy-maker

Consider an activist policy-maker, in the economy described by Equations (4) and (5) of Section 2, facing an asset-price bubble developing according to Equation (3), and seeking to minimise the loss function given by Equation (6). As in Section 3, we assume that the bubble will burst with certainty in year 14, if it has not already done so. We also assume it to be unaffected by the actions of policy-makers.[23]

For each period $s = 0, 1, \ldots, 13$ we wish to derive the *analytic solution* to the policy-maker's problem of determining the optimal current policy response to this situation. Note that, since the bubble will definitely have burst by period 14 (and is assumed not to re-form thereafter), we already know what an activist will recommend for policy in periods $s = 14, 15, \ldots$, namely that policy simply be given by Equation (7).

In formulating the optimal policy problem facing an activist policy-maker in each period $s = 0, 1, \ldots, 13$, it is crucial to appreciate that we seek here the *contingent* policy recommendation which such a policy-maker would make. In other words, we seek the recommendation they would make on the understanding that, whenever the bubble does ultimately burst, policy may be switched from then on to a profile better suited to an economy no longer experiencing an asset-price bubble.[24] Specifically, for each period $s = 0, 1, \ldots, 13$, we therefore seek the optimal $(14 - s)$-component vector, R_s^{ac}, of contingent policy recommendations which an activist would, in period s, wish to see enacted in periods $\{s, s + 1, \ldots, 13\}$ *unless the bubble bursts*, say in period $s + k$, in which case policy, for periods $\{s + k, s + k + 1, \ldots, 13\}$, would then switch to being set by Equation (7). The activist's actual policy recommendation for period s, r_s^{ac}, is then just the first component of this vector R_s^{ac}.

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[23] This assumption applies throughout this Appendix. Analytical treatment of bubbles whose development is affected by the actions of policy-makers turns out to be far more difficult than that of bubbles which are unaffected by policy. Moreover, for bubbles which are influenced by policy, analytical treatment does not yield a simple, closed formula for the difference between the policy recommendations of activist and sceptical policy-makers. This contrasts with the results obtained in this Appendix for bubbles which are uninfluenced by policy.

[24] See Footnote 6 regarding the contingent nature of the policy problem facing an activist policy-maker in the current setting.

C.2 A Convenient Matrix Form for the Solution of the Ball Model

We begin by writing the Ball model in a slightly more general form than that used in the main body of the paper, *viz*:

$$\begin{aligned} y_t &= -\beta r_{t-1} + \lambda y_{t-1} + u_t \\ \pi_t &= \pi_{t-1} + \alpha y_{t-1} \,. \end{aligned} \tag{C1}$$

Here we have included a term for a general exogenous shock in period t, u_t, in the output gap equation. Writing this equation in this way helps to simplify the subsequent discussion – even though we ultimately wish to focus on the case where the shocks $\{u_t\}$ arise from a bubble via $u_t \equiv \Delta a_t$, as in Equation (4).

Now consider any fixed period $s \in \{0,1,\ldots,13\}$. To determine an activist policy-maker's policy recommendation for period s, r_s^{ac}, we need first to establish a suitable matrix form for the solution of the Ball model, Equation (C1), over the horizon $\{s+1, s+2, \ldots, 14\}$, in the event that: the economy is expected to be struck by some given set of exogenous shocks $\{u_{s+1}, \ldots, u_{14}\}$; and that policy in periods $\{s, s+1, \ldots, 13\}$ is to be set according to some given path $\{r_s, r_{s+1}, \ldots, r_{13}\}$. By solution of the Ball model we mean here determination of the profiles for output and inflation over the horizon $\{s+1, s+2, \ldots, 14\}$.

To this end, set $N_s \equiv 14 - s$. Then, for any $t \leq s$, let Y_t, Π_t, R_t and X_t denote the $N_s \times 1$ vectors $Y_t = (y_{t+1}, \ldots, y_{t+N_s})^T$, $\Pi_t = (\pi_{t+1}, \ldots, \pi_{t+N_s})^T$, $R_t = (r_t, \ldots, r_{t+N_s-1})^T$ and $X_t = (u_{t+1}, \ldots, u_{t+N_s})^T$. Also, let A denote the $2N_s \times 2N_s$ matrix, and Z_t and ξ_t the $2N_s \times 1$ vectors, given by

$$A \equiv \begin{pmatrix} \lambda I_{N_s} & 0 \\ \alpha I_{N_s} & I_{N_s} \end{pmatrix} \quad , \quad Z_t \equiv \begin{pmatrix} Y_t \\ \Pi_t \end{pmatrix} \quad , \quad \xi_t \equiv \begin{pmatrix} X_t - \beta R_t \\ 0 \end{pmatrix} \tag{C2}$$

where I_{N_s} denotes the $N_s \times N_s$ identity matrix. Finally, let H denote the $2N_s \times N_s$ matrix given by $H = (I_{N_s} \;\; 0)^T$, so that $\xi_t = H(X_t - \beta R_t)$.

Then, for any $t \leq s$ we have that Ball's model for the N_s-period horizon $\{t+1, \ldots, t+N_s\}$ may be written compactly in matrix form as the relationship $Z_t = AZ_{t-1} + \xi_t$. By simple iteration (and writing A^0 for I_{2N_s}) it then follows that

$$Z_s = A^{N_s} Z_{s-N_s} + \sum_{j=0}^{N_s-1} A^j \xi_{s-j} \,. \tag{C3}$$

This represents an expression for Z_s in terms of:

- future exogenous shocks $\{u_t\}_{t=s+1}^{14}$, and current and future policy settings $\{r_t\}_{t=s}^{13}$, which enter through the vectors $\{\xi_t\}_{t=s-N_s+1}^{s}$; together with
- initial conditions for the endogenous variables y_t and π_t, as captured by the vector Z_{s-N_s}.

Moreover, using that $\xi_t = H(X_t - \beta R_t)$, it follows immediately that we may write

$$Z_s = \left\{ A^{N_s} Z_{s-N_s} + \sum_{j=0}^{N_s-1} A^j H X_{s-j} \right\} - \beta \sum_{j=0}^{N_s-1} A^j H R_{s-j} \,. \tag{C4}$$

To proceed further, now introduce the 'backward' and 'forward' shift operators, B and F, given in matrix form by the $N_s \times N_s$ matrices

$$B \equiv \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad , \quad F \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} . \tag{C5}$$

Then observe that, for any $j = 0, \ldots, N_s - 1$, we may write

$$X_{s-j} = B^j X_s + F^{N_s - j} X_{s-N_s} \quad , \quad R_{s-j} = B^j R_s + F^{N_s - j} R_{s-N_s} \,. \tag{C6}$$

Hence we may rewrite Equation (C4) in the form

$$\begin{aligned} Z_s = &\left\{ A^{N_s} Z_{s-N_s} + \textstyle\sum_{j=0}^{N_s-1} A^j H F^{N_s-j} X_{s-N_s} - \beta \textstyle\sum_{j=0}^{N_s-1} A^j H F^{N_s-j} R_{s-N_s} \right\} \\ &+ \left\{ \textstyle\sum_{j=0}^{N_s-1} A^j H B^j X_s \right\} - \left\{ \beta \textstyle\sum_{j=0}^{N_s-1} A^j H B^j R_s \right\} \end{aligned} \tag{C7}$$

where the first bracket of terms here represents purely 'historical' effects (that is, the influence of initial conditions); the second captures the impact of expected exogenous shocks over the horizon under consideration; and the third reflects the influence of monetary policy settings over this same horizon.

For any given initial conditions, given set of policy decisions $\{r_s, \ldots, r_{13}\}$, and given set of exogenous shocks $\{u_{s+1}, \ldots, u_{14}\}$, Equation (C7) represents the general solution to the Ball model, over the horizon $\{s+1,\ldots,14\}$, expressed in suitable matrix form. Moreover, if we now introduce the notation

$$J_s \equiv A^{N_s} Z_{s-N_s} + \sum_{j=0}^{N_s-1} A^j H F^{N_s-j} X_{s-N_s} - \beta \sum_{j=0}^{N_s-1} A^j H F^{N_s-j} R_{s-N_s} \tag{C8}$$

and then set

$$K \equiv \sum_{j=0}^{N_s-1} A^j H B^j \quad , \quad G_s = J_s + K X_s \tag{C9}$$

then we may write Equation (C7) more compactly as the formula

$$Z_s = P R_s + G_s \tag{C10}$$

where P is the $2N_s \times N_s$ matrix defined by $P \equiv -\beta K$.

C.3 Mathematical Formulation of the Problem Facing an Activist Policy-maker in Period s

We now wish to formulate precisely the optimal policy problem facing an activist policy-maker in period s, trying to determine the optimal policy vector, R_s^{ac}, of *contingent policy settings* to recommend for periods $\{s, s+1,\ldots,13\}$. To do this, it is necessary first to introduce yet some further notation.

First, let $\{X_s^{(k)}\}_{k=1}^{N_s}$ denote the N_s different possible $N_s \times 1$ vectors of exogenous shocks which an activist policy-maker might expect to hit the economy in periods $\{s+1, s+2,\ldots,14\}$, depending (respectively) on whether the bubble bursts in period $s+1$, period $s+2$, ..., or period 14. Also, let $\{p_s^{(k)}\}_{k=1}^{N_s}$ denote the associated probabilities with which each of these possible shock profiles is expected to occur, as at period s.[25] Note that, in view of Equation (3) of the

[25] Note that these probabilities are not the same as the probabilities, $\{p_t\}$, referred to in the main body of the paper. Recall that, for each t, p_{t+1} denotes the conditional probability that the bubble will burst in period $t+1$, given that it has not done so by period t. For any period s, the two sets of probabilities are thus clearly related by the formulae: $p_s^{(1)} = p_{s+1}$; $p_s^{(2)} = (1-p_{s+1})p_{s+2}$; $p_s^{(3)} = (1-p_{s+1})(1-p_{s+2})p_{s+3}$; and so on.

main paper for the way an asset-price bubble evolves, the vectors $X_s^{(k)}$ are given explicitly, for each $k = 1, \ldots, N_s$, by the formula

$$X_s^{(k)}(j) = \begin{cases} \gamma_{s+j} & , \quad 1 \leq j < k \\ -\sum_{i=1}^{s+k-1} \gamma_i & , \quad j = k \\ 0 & , \quad k < j \leq N_s \,. \end{cases} \tag{C11}$$

Next, for any given choice of vector of contingent policy recommendations by the activist policy-maker, R_s^{ac}, let $\{R_s^{(k)}\}_{k=1}^{N_s}$ denote the corresponding set of policy paths which would *actually be followed* by the activist over the horizon $\{s, s+1, \ldots, 13\}$, depending upon when the bubble actually bursts. Thus, for each k, $R_s^{(k)}$ is an $N_s \times 1$ vector whose first k entries would be the same as those of R_s^{ac}, but whose remaining entries would then be as determined by Equation (7) (see also Equation (D2) of Appendix D).

Finally, let $\{Z_s^{(k)}\}_{k=1}^{N_s}$ denote the corresponding $2N_s \times 1$ vectors of outcomes for output and inflation over the horizon $\{s+1, \ldots, 14\}$ which would occur in the event that the bubble bursts (respectively) in period $s+1$, period $s+2$, ..., or period 14. Thus, $Z_s^{(k)}$ denotes the vector of outcomes for output and inflation which would occur in the event that $X_s^{(k)}$ describes the exogenous shocks striking the economy over this horizon, and that policy is given by the vector $R_s^{(k)}$. Note that, by Equation (C10), we will then have simply, for each k:

$$Z_s^{(k)} = PR_s^{(k)} + G_s^{(k)} \quad \text{where} \quad G_s^{(k)} = J_s + KX_s^{(k)} \,. \tag{C12}$$

Armed with this notation, we can now formulate precisely the optimal policy problem facing an activist policy-maker in period s. For now, invoking also Result 1 of Appendix D, it is clear that, for any given choice of contingent policy recommendations R_s^{ac} over the horizon $\{s, s+1, \ldots, 13\}$, the corresponding loss expected by an activist policy-maker would be

$$L = \sum_{\tau=s+1}^{\infty} \left[E_s(y_\tau^2) + \mu E_s(\pi_\tau^2) \right] = \sum_{k=1}^{N_s} p_s^{(k)} L_s^{(k)} \tag{C13}$$

where

$$L_s^{(k)} = {Z_s^{(k)}}^T \Omega_s^{(k)} Z_s^{(k)} \tag{C14}$$

and where the $2N_s \times 2N_s$ matrices $\{\Omega_s^{(k)}\}_{k=1}^{N_s}$ are given by

$$\left(\Omega_s^{(k)}\right)_{ij} = \begin{cases} 1 & , \ 1 \leq i = j < k \\ 1+\zeta\alpha^2 & , \ i = k, j = k \\ \zeta\alpha & , \ i = k, j = k+N_s \\ \mu & , \ 1+N_s \leq i = j < k+N_s \\ \zeta\alpha & , \ i = k+N_s, j = k \\ \mu+\zeta & , \ i = k+N_s, j = k+N_s \\ 0 & , \ \text{otherwise}\,. \end{cases} \tag{C15}$$

Here, $\zeta = (\mu+q^2)/(1-(1-\alpha q)^2)$ is a scalar which arises from the working in Appendix D, expressed in terms of another scalar, $q = \left(-\mu\alpha + (\mu^2\alpha^2 + 4\mu)^{1/2}\right)/2$.

Therefore, finally, in any period $s = 0, 1, \ldots, 13$, the activist policy-maker's task of finding the optimal contingent set of policy settings to recommend over the horizon $\{s, s+1, \ldots, 13\}$ may be expressed succinctly as: find the policy vector R_s^{ac} which minimises

$$L = \sum_{k=1}^{N_s} p_s^{(k)} L_s^{(k)} \tag{C16}$$

where each $L_s^{(k)}$ is as given by Equation (C14), subject to the condition that

$$Z_s^{(k)} = PR_s^{(k)} + G_s^{(k)} \tag{C17}$$

for each k (where the quantities P, $R_s^{(k)}$ and $G_s^{(k)}$ are as defined earlier).

C.4 Solution of the Problem Facing an Activist Policy-maker

To solve the optimal policy problem just posed, observe first that, for each $k = 1, \ldots, N_s$, by putting Equation (C17) into Equation (C14) and expanding we may write

$$\begin{aligned} L_s^{(k)} &= \left(PR_s^{(k)} + G_s^{(k)}\right)^T \Omega_s^{(k)} \left(PR_s^{(k)} + G_s^{(k)}\right) \\ &= R_s^{(k)T} P^T \Omega_s^{(k)} PR_s^{(k)} + 2R_s^{(k)T} P^T \Omega_s^{(k)} G_s^{(k)} + G_s^{(k)T} \Omega_s^{(k)} G_s^{(k)}\,. \end{aligned} \tag{C18}$$

To simplify this expression we next exploit the fact that, for each k, the vectors $R_s^{(k)}$ and R_s^{ac} have the same first k components. One implication of this is that, for each k, we have the identity

$$\Omega_s^{(k)} P R_s^{(k)} = \Omega_s^{(k)} P R_s^{ac} . \tag{C19}$$

Hence, noting also that $\Omega_s^{(k)}$ is a symmetric matrix, we may rewrite Equation (C18) as

$$L_s^{(k)} = R_s^{acT} P^T \Omega_s^{(k)} P R_s^{ac} + 2 R_s^{acT} P^T \Omega_s^{(k)} G_s^{(k)} + {G_s^{(k)}}^T \Omega_s^{(k)} G_s^{(k)} \tag{C20}$$

for each $k = 1, \ldots, N_s$. Then, in view of Equation (C16), it follows that

$$\begin{aligned} L \;=\; & R_s^{acT} P^T \left(\sum_{k=1}^{N_s} p_s^{(k)} \Omega_s^{(k)} \right) P R_s^{ac} + \\ & 2 R_s^{acT} P^T \left(\sum_{k=1}^{N_s} p_s^{(k)} \Omega_s^{(k)} G_s^{(k)} \right) + \sum_{k=1}^{N_s} p_s^{(k)} {G_s^{(k)}}^T \Omega_s^{(k)} G_s^{(k)} \end{aligned} \tag{C21}$$

and the activist policy-maker's task is to choose R_s^{ac} so as to minimise this quantity. Yet the solution to this optimisation problem is well-known to be given by

$$R_s^{ac} = - \left(P^T \left(\sum_{k=1}^{N_s} p_s^{(k)} \Omega_s^{(k)} \right) P \right)^{-1} P^T \left(\sum_{k=1}^{N_s} p_s^{(k)} \Omega_s^{(k)} G_s^{(k)} \right) . \tag{C22}$$

Finally, this expression may be simplified slightly if we introduce the notation

$$\Omega_s = \sum_{k=1}^{N_s} p_s^{(k)} \Omega_s^{(k)} \quad , \quad \chi_s = \sum_{k=1}^{N_s} p_s^{(k)} \Omega_s^{(k)} P X_s^{(k)} . \tag{C23}$$

Then, using the definitions of P and $G_s^{(k)}$, it follows from Equation (C22) that

$$R_s^{ac} = - \left(P^T \Omega_s P \right)^{-1} P^T \Omega_s J_s + \beta^{-1} \left(P^T \Omega_s P \right)^{-1} P^T \chi_s \tag{C24}$$

which expresses R_s^{ac} as a function of the matrices P and Ω_s, the vectors J_s and χ_s, and the parameter β.

Note that, importantly, the vector J_s in Equation (C24) is a function purely of the vectors Z_{s-N_s}, X_{s-N_s} and R_{s-N_s}. Hence, the solution vector, R_s^{ac}, given by Equation (C24) may be viewed as consisting of two parts: a part reflecting the *past* set of economic outcomes, policy actions and exogenous shocks which have occurred up to period s, captured in the first term on the right hand side of Equation (C24); and a part reflecting the *future* exogenous shocks which the policy-maker expects to buffet the economy over the policy horizon $\{s+1, s+2, \ldots, 14\}$, as captured by the second right hand side term of Equation (C24).

C.5 The Difference between the Recommendations of Activists and Sceptics

Fortunately, we are really only concerned with the first component of the vector R_s^{ac}, since this is the policy recommendation which the activist must actually make for the current quarter. Denote this first component by r_s^{ac}, and write r_s^{sc} for the corresponding policy recommendation of a sceptic for the current quarter. Then it turns out that, using a matrix algebra result set out in Appendix D, we can derive from Equation (C24) a simple analytical expression for the quantity $(r_s^{ac} - r_s^{sc})$, the *difference* between the recommendations in each quarter of activist and sceptical policy-makers.

In more detail, one way to think about a sceptic is as *an activist who thinks that all the vectors $X_s^{(k)}$ are zero, and so expects no future exogenous shocks to output*. It follows that the policy recommendation of a sceptic, in each period, is also given by the first component of a vector of the form given by Equation (C24) – except with the quantity 'χ_s' treated as being zero, so that the second term in this formula vanishes. Hence, r_s^{sc} is given, in each period s, by the first component of the vector $\left(P^T \Omega_s P\right)^{-1} P^T \Omega_s J_s$, where the vector J_s is the same as for an activist policy-maker.[26]

[26] Here we are using that our activist and sceptical policy-makers face an economy in the same state in each period s, which ensures that the vector J_s is the same for both types of policy-makers.

This, however, then means that the *difference* between the policy recommendations of an activist and a sceptic in each period s will be given simply by the first component of the vector

$$\beta^{-1}\left(P^T\Omega_s P\right)^{-1}P^T\chi_s\,. \tag{C25}$$

Finally, this then turns out to yield a simple formula for this *difference* between the policy recommendations of activist and sceptical policy-makers, in view of Result 2 in Appendix D. We obtain that this difference is independent of the model parameters α and λ, and of the loss function parameter μ. Explicitly it depends, in each period s, only upon the bubble's expected growth next period if it survives, γ_{s+1}, its current size $a_s \equiv \sum_{j=1}^{s}\gamma_j$, and the probability, $p_{s+1} \equiv p_s^{(1)}$, that it will burst in period $s+1$, given that it has not done so by period s:

$$r_s^{ac} - r_s^{sc} = \beta^{-1}\left\{(1-p_{s+1})\gamma_{s+1} - p_{s+1}a_s\right\} \tag{C26}$$

which is precisely the formula noted in Footnote 6 in Section 3.

Appendix D: Some Technical Results Required in Appendix C

In this Appendix we set out a number of technical results which we called upon but did not justify in Appendix C, so as not to interrupt the flow of the discussion.

D.1 How the Infinite Loss Horizon is Handled in Appendix C

Recall that the loss function used in the main body of the paper is

$$L = \sum_{\tau=s+1}^{\infty} \left[E_s(y_\tau^2) + \mu E_s(\pi_\tau^2) \right] . \tag{D1}$$

In each period s, this combines contributions from each period of the *infinite* horizon $\{s+1, s+2, \dots\}$. In Appendix C, however, this loss is computed using *finite dimensional* matrix algebra involving the matrices $\{\Omega_s^{(k)}\}_{k=1}^{N_s}$.

To understand how this is done, the key observation is that, once a bubble has burst, no further shocks are expected to hit the economy. Then, if the bubble bursts in period $s+k$, it is well known (see (Ball 1999)) that, to minimise L, the optimal setting for policy, in period $s+k$ and all subsequent periods $\{s+k+1, s+k+2, \dots\}$, is given recursively by

$$r_{s+j} = \beta^{-1}(\lambda + \alpha q) y_{s+j} + \beta^{-1} q \pi_{s+j} \tag{D2}$$

where the scalar q is defined by $q = \left(-\mu\alpha + (\mu^2\alpha^2 + 4\mu)^{1/2}\right)/2$. Note that Equation (7) is just a special case of this general formula, for the case $\lambda = 0.8$, $\alpha = 0.4$, $\beta = 1$ and $\mu = 1$.

Using this, it turns out that, for the case of a bubble which bursts in period $s+k$, it is possible to express the total contribution to the loss function L, from all periods $t > s+k$, purely in terms of the values of y and π in period $s+k$. The precise result, the proof of which is available from the authors upon request, is as follows.

Result 1. *Consider an activist policy-maker in period s, facing an asset-price bubble which is expected to burst in period $s+k$, after which no further exogenous shocks are expected to strike the economy. Then the quantity*

$$L^* = \sum_{\tau=s+k+1}^{\infty} \left[E_s(y_\tau^2) + \mu E_s(\pi_\tau^2) \right] \tag{D3}$$

satisfies that

$$L^* \geq \frac{\mu + q^2}{1-(1-\alpha q)^2}(\alpha y_{s+k} + \pi_{s+k})^2 \tag{D4}$$

with equality if and only if policy, in all periods $t \geq s+k$, is set according to the recursive rule given by Equation (D2).

It is now easy to see how the use of the infinite horizon loss function, Equation (D1), is accommodated within the theoretical framework of finite dimensional matrix algebra used in Appendix C. For consider an activist policy-maker in period s facing an asset-price bubble which has not yet burst. As in Appendix C, let $L_s^{(k)}$ denote the contingent loss such a policy-maker would expect were the bubble expected to burst in period $s+k$. Then, in view of Result 1, we clearly have that in this setting, and with policy for $t \geq s+k$ set according to Equation (D2),

$$\begin{aligned} L_s^{(k)} &= \sum_{\tau=s+1}^{\infty} \left[E_s(y_\tau^2) + \mu E_s(\pi_\tau^2)\right] \\ &= \frac{\mu + q^2}{1-(1-\alpha q)^2}(\alpha y_{s+k} + \pi_{s+k})^2 + \sum_{\tau=s+1}^{s+k} \left[E_s(y_\tau^2) + \mu E_s(\pi_\tau^2)\right] \\ &= Z_s^{(k)^T} \Omega_s^{(k)} Z_s^{(k)} \end{aligned} \tag{D5}$$

where $Z_s^{(k)}$ and $\Omega_s^{(k)}$ are as defined in Section C.3 of Appendix C; and this completes the justification of Equations (C13) and (C14).

D.2 A Matrix Algebra Result

In Section C.5 of Appendix C we also invoked the following linear algebra result.

Result 2. *For any period s let the matrices P and Ω_s, and the vector χ_s, be as defined in Appendix C. Next, let V denote the $N_s \times 1$ vector*

$$V = \left(P^T \Omega_s P\right)^{-1} P^T \chi_s\,. \tag{D6}$$

Then there is a simple formula for $V(1)$, the first component of this vector V, in terms of: the bubble's expected growth next period if it survives, γ_{s+1}; its current *size* $a_s \equiv \sum_{j=1}^{s} \gamma_j$*; and the probability,* $p_{s+1} \equiv p_s^{(1)}$*, that it will burst in period* $s+1$*, given that it has not done so by period* s*. This formula is*

$$V(1) = (1 - p_{s+1})\gamma_{s+1} - p_{s+1} a_s\,. \tag{D7}$$

Proof: The proof of this result is quite lengthy and technical. It is available from the authors upon request. The basic idea, however, is first to establish that we may re-write the vector V in the form

$$V = X_s^{(N_s)} - \beta \sum_{k=1}^{N_s-1} \sigma_k W_s^{(k)} \tag{D8}$$

where $X_s^{(N_s)}$ is as given by Equation (C11), and where, for each $k = 1, \ldots, N_s - 1$: σ_k denotes the quantity $\sigma_k \equiv \sum_{j=1}^{s+k} \gamma_j$; e_k denotes the $N_s \times 1$ vector $(0, \ldots, 0, 1, 0, \ldots, 0)^T$, the '1' appearing in the k^{th} entry; and $W_s^{(k)}$ denotes the vector

$$W_s^{(k)} \equiv \beta^{-1} p_s^{(k)} \left(P^T \Omega_s P\right)^{-1} P^T \Omega_s^{(k)} P e_k\,. \tag{D9}$$

It then follows directly from Equation (D8) that, to obtain Equation (D7), it will suffice to prove the general formula that, for any $k = 1, \ldots, N_s - 1$:

$$W_s^{(k)}(1) = \begin{cases} \beta^{-1} p_s^{(1)} & , \quad k = 1 \\ 0 & , \quad \text{otherwise}\,. \end{cases} \tag{D10}$$

Finally, this latter result may be established by: first re-casting the vector quantities $W_s^{(k)}$ as arising from a *loss minimisation problem*, analogous to the way that the vector V did in Appendix C; and then making a judicious 'change of variables' with respect to which to carry out the loss minimisation – a change of variables prompted by the structure of the Ball model.[27] QED

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[27] Specifically, recall that in the Ball model changes in interest rates flow through to inflation (via output) with a lag of two periods. Hence, the following two options are readily seen to be equivalent: on the one hand, determining an optimum profile for interest rates, r_t, in periods $t = s, s+1, \ldots, 13$, so as to minimise a given loss function; and on the other, seeking instead an optimum profile for *inflation*, π_t, in periods $t = s+2, s+3, \ldots, 15$, so as to minimise this loss, and then recursively back-solving for the corresponding implied profile for $\{r_t\}_{t=s}^{13}$. It is this latter approach which we employ.

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