EQUILIBRIUM EXCHANGE RATES AND A POPULAR MODEL OF INTERNATIONAL ASSET DEMANDS: AN INCONSISTENCY

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ABSTRACT

Many of the continuous time Ito process models of international asset demand deal with the simplest case of geometric Brownian motion processes for asset prices and exchange rates. Following the single country models, these international models yield restrictions on the moments of the price processes via the solution of asset market clearing conditions. This produces an International Capital Asset Pricing model, but it does not also deliver restrictions on the exchange rate processes. This paper shows that consideration of associated equilibrium conditions in the foreign exchange markets (which are inherent in the international version of the model) produces such restrictions, allowing full pricing of assets in the various currencies and exchange rates. However, it is also shown that the assumption of geometric Brownian motion for exchange rates is inconsistent with these restrictions. This suggests the need for further work to impose the additional equilibrium constraints in models with more flexible price processes.

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1. Introduction

Adler and Dumas (1983) and Branson and Henderson (1985) have surveyed the applications to the international setting of Merton's (1969 and 1971) seminal work applying the tools of stochastic calculus to the analysis of the micro-foundations of asset demands. The distinguishing characteristic of these international models is that the real return on a given asset is not perceived to be identical by agents in different countries - i.e., real exchange rates differ (stochastically) across countries. Most of these studies have dealt with the simplest case of geometric Brownian motion for asset prices and exchange rates.[1] The analyses have typically proceeded in one of two broad directions.

Firstly, much attention has been paid to the form of individual asset demand equations under different assumptions about the menu of assets available, the currency of denomination, the nature of exchange rate and general price level risk, and the links between these risks and those coming directly from asset prices. The main results to emerge from this work have concerned portfolio separations that arise when foreign assets and their associated risks are introduced into the model, and the larger number of mutual funds that are required to span the investment opportunity space when it is perceived heterogeneously by different agents - i.e., a full set of mutual funds is typically required in each country.[2]

The other branch of analysis has involved using the partial equilibrium constraints imposed by market clearing on the parameters of the asset price processes to develop an International Capital Asset Pricing

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1. This paper does not deal directly with the work of Hodrick (1981), Stulz (1981) or others who follow Merton (1973) or Breeden (1979) in postulating more complicated processes for asset prices. It does conclude that this extension is a necessary (but not sufficient) requirement for a consistent model in the international setting.

2. These results are discussed in both Adler and Dumas (1983) and Branson and Henderson (1985).

Model.[3] For example, in their survey Adler and Dumas (1983) solve for the equilibrium relationships between the instantaneous expected returns on the assets under the assumption that the supplies of bonds and other securities are fixed. The results here have been more limited. At best, these models are capable of pricing the assets only when the exchange rate parameters are given. These models appear to be a translation to many countries of the earlier Capital Asset Pricing Model results in that one "security" (in effect the exchange rate) remains unpriced in each country. At worst, the restrictions obtained are essentially non-testible due to unobservability problems that arise form the aggregation of demands by agents who perceive real returns heterogeneously.

Attempts at addressing these issues often bypass the aggregation problems by ignoring the heterogeneity of the purchasing power of different monies, either by assuming purchasing power parity or a global numeraire. However, this is the very property that seems to distinguish international finance from its closed economy counterpart.

All these models assume market clearing, yet focus solely on clearing in asset markets and asset price determination in terms of a single currency; they ignore the fundamental issue of exchange rate determination. Endogenising the exchange rate would allow assets to be priced in terms of every currency, by pricing the currencies themselves. The central contribution of this paper is the explicit inclusion of equilibrium conditions for foreign exchange markets (balance of payments equilibrium).

Applying this analysis to the model which is commonly encountered in the international literature reveals a fundamental flaw. The assumption of geometric Brownian motion for exchange rates is inconsistent with the constraints imposed by balance of payments equilibrium.[4] Moreover, real exchange rates do not posses idiosyncratic risk - instantaneous exchange rate risk is no more than a combination of the risks on individual asset prices.

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3. The analysis in this paper, as in almost all the associated literature, is partial equilibrium. In particular, it takes asset supply behaviour as given. The general equilibrium work of Cox, Ingersoll and Ross (1985a) and (1985b) remains to be extended to an international setting. Such an extension will require the explicit treatment of multi-good economies.

4. Rosenburg and Ohlson (1976) point out a problem with the assumption of geometric Brownian motion for asset prices in the domestic setting.

The remainder of the paper is set out in six sections. Section 2 presents the model of individual agents' asset demands and the aggregate market clearing conditions in fairly general terms and outlines the proof methodology that is used in the following sections. The two sections that follow illustrate the inconsistency of the typically assumed exchange rate process with the restrictions derived from the equilibrium conditions. These examples both involve a single asset in each of two countries. The more general case is presented in Sections 5 and 6. Concluding comments may be found in Section 7. The continuous time budget constraints and equilibrium conditions are derived from discrete time in the Appendix.

2. The Model

2.1 Assumptions

Models that fall within the class under examination typically assume that agents: have incomes that are derived solely from capital gains; have rational expectations;[5] and act as price takers, believing that they can buy and sell as much of any asset as they like at the market price. Markets are structured such that they are always in equilibrium;[6] there are no transactions costs, taxes or problems with indivisibilities of assets or goods; borrowing and lending can be done at the same rate of interest; and there are no restrictions on short sales of any asset. The real or supply side of the asset market is assumed to be such that the real return on any asset is given by a geometric Brownian motion process.[7]

In addition to these standard assumptions, it will be assumed without loss of generality that the world consists of only two countries, each with one infinitely-lived representative agent.[8] (A "hat" will be used to denote

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5. That is, their subjective expectations are exactly statistical expectations conditional on available information. While this assumption may not be explicit in some of the models, it is always implicitly used in solving the agents' optimisation problem.

6. More precisely, there is no trading at non-equilibrium prices.

7. The assumption of these Ito processes means that functions in the model will generally be right-continuous but not differentiable in the usual sense. Instead the tools of stochastic calculus will be used. In particular, care needs to be taken in specifying the form of budget constraints and equilibrium conditions to ensure that they are consistent with the properties of Ito processes. See Merton (1971) for references to the technical literature on Ito processes and stochastic calculus.

8. This assumption is purely one of notational convenience. It has no bearing, other than simplification, on the results to follow.

the foreign agent/country where a distinction is required.) Assume also that there exist two general price levels, $Q(t)$ in the home country and $\hat{Q}(t)$ in the foreign country; H consumption goods with nominal prices $Q_i(t)$ faced by the home agent $\hat{Q}_i(t)$ faced by the foreign agent. There are S distinct financial assets with nominal prices $P_j(t)$ and $\hat{P}_j(t)$, as perceived by the home and foreign agent respectively.[9]

Under these conditions agents will trade continuously in time. They will choose their instantaneous real rates of flow of consumption goods, $c_i(t)$'s, and their asset stocks, $n_j(t)$'s, to maximise the expected discounted present value of lifetime utility. Following Adler and Dumas (1983), assume that each agent's instantaneous utility function, $U(c_1(t),\ldots,c_H(t))$, is homothetic.[10] The instantaneous indirect utility function will then be $V(C(t),Q(t))$ where the first argument, $C(t) \equiv \Sigma_i c_i(t)Q_i(t)$, is the instantaneous rate of total consumption and the second argument, $Q(t) \equiv Q(Q_1(t),\ldots,Q_H(t))$, is a price index rather than a price vector as in the general case.[11] The function $V(.)$ is known to be homogenous of degree zero in C and Q. Hence, it may be transformed to a function of one variable, $V(c(t)) \equiv V(c(t),1)$, where $c(t) \equiv C(t)/Q(t) = \Sigma_i c_i(t)q_i(t)$, is the instantaneous rate of total real consumption expenditure and $q_i(t) \equiv Q_i(t)/Q(t)$ is the real (deflated) price of the i^{th} consumption good.[12]

The assumption of homotheticity allows an agent's optimisation problem to be treated as a multistage budgeting problem. There are two stages. The first is the division of income (capital gains) between a consumption budget and a savings budget. At the second stage the savings

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9. Distinct in the sense that <u>all</u> of the S assets are required to span the investment opportunity set as it is perceived by either agent.

10. The effect of this and other assumptions on the results will be addressed later in the paper.

11. Homotheticity produces two restrictions. The first is that $Q(.)$ is homogenous of degree one in the $Q_i(t)$'s. The second is that $V(.)$ is monotone increasing. This latter condition is required (at least in a weak form) of any indirect utility function.

12. Adler and Dumas (1983) follow much of the international literature and work in nominal terms. They use the homogeneity of $V(.)$ to obtain restrictions on its partial derivatives (with respect to nominal wealth and the price level) rather than to define a new function. The method used here is more explicit in dealing with the absence of money illusion in functions that are later derived from $V(.)$ (e.g., the homogeneity of degree zero in nominal wealth and the price level of the function $\Theta(.)$). It may readily be verified that both methods produce the same results.

budget is allocated across available assets and the consumption budget is allocated across consumption goods. These second stage allocation decisions are independent of each other. The latter one is not important for what follows and may be dropped from the analysis.

2.2 Derivation of Demand Functions[13]

The stochastic processes for nominal asset prices, the general price levels and the nominal exchange rate have been assumed such that all real (i.e., deflated) assets returns are given by geometric Brownian motion processes,

$$(1) \qquad dp_j(t)/p_j(t) = \mu_j dt + \sigma_j dz_j \qquad j=1,\ldots S$$

where $p_j(t) \equiv P_j(t)/Q(t)$ is the deflated asset price. The μ_j's and (every element of the) σ_j's are constant, and the z_j's are (perhaps vector) components of a multivariate Wiener process with (every element of) dz_j having a constant mean of zero, variance of unity and constant covariance (matrices) denoted by $E_t(dz_i dz_j')=\rho_{ij}$, where E_t is the statistical expectation conditional on information available at t. For convenience, denote (scalar) terms such as $\sigma_i \rho_{ij} \sigma_j = \text{cov}_t[(dp_i/p_i),(dp_j p_j)]$, by σ_{ij}.

Many of the previous studies dealing with international asset demands have assumed returns to be denominated in nominal units. In that case, general price level risk is an important determinant of asset demands. The specification adopted here allows for this as a special case.[14] However, since agents are only interested in expected real returns and real risks, it will simplify the expressions that follow if these nominal effects are subsumed into the compact notation of real returns.

Equation (1) may be used to obtain the "budget constraint" (Ito process for real wealth),[15]

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13. The demand functions for the home country agent are derived here; those for the foreign agent may be obtained simply by placing a hat over the variables.

14. For instance, one could define $\mu_j \equiv (\bar{\mu}_j - \mu_Q + \sigma_Q^2 - \bar{\sigma}_{jQ})$ (a scalar),

    $\sigma_j \equiv [\bar{\sigma}_j, -\sigma_Q]$ (a 1x2 vector) and $dz_j \equiv [d\bar{z}_j, dz_Q]'$ (a 2x1 vector),

    where the bar indicates the nominal return on an asset and the subscript Q indicates the parameters of the process for the general price level.

15. This equation incorporates both a "stock" and "flow" budget constraint. See the Appendix for a derivation from discrete time.

(2) $$dw(t) = w(t)[\{\mu_1 + \Sigma_j a_j(t)(\mu_j - \mu_1) - c(t)/w(t)]dt + w(t)[\sigma_1 dz_1 + \Sigma_j a_j(t)(\sigma_j dz_j - \sigma_1 dz_1)]$$

where the summation over j runs from 2 to S and the a_j's are the shares of wealth invested in the last (S-1) assets,

(3) $$a_j(t) \equiv n_j(t)p_j(t)/w(t) \qquad j = 2,\ldots,S$$

Subject to this budget constraint, initial conditions and a transversality condition, agents will choose their instantaneous rate of total real consumption and their (S-1) independent asset shares to maximise,

$$E_t \int_t^{\infty} e^{-\delta\tau} V(c(\tau))d\tau \qquad t\epsilon[0,\infty)$$

Let $e^{-\delta t}J(w(t))$ be this maximum. Then by the Bellman "Principle of Optimality",

(4) $$0 = \max \{e^{-\delta t}V(c(t)) + E_t\ d[e^{-\delta t}J(w(t))]/dt$$

where, from Ito's Lemma, the evaluation of the Dynkin operator on the maximum function is,

$$E_t d[e^{-\delta t}J]/dt = e^{-\delta t}[-\delta J + E_t(dw)J' + {}^1/_2 var_t(dw)J"]$$

with the superscripts on J(.) indicating derivatives.[16] From the budget constraint, equation (2), one may obtain,

$$E_t(dw) = w[\mu_1 + \Sigma_j a_j(\mu_j - \mu_1)] - c$$

$$var_t(dw) = w^2[\sigma_1^2 + 2\Sigma_j a_j(\sigma_{j1} - \sigma_1^2) + \Sigma_j\Sigma_i a_j a_i(\sigma_{ji} - \sigma_{j1} - \sigma_{1i} + \sigma_1^2)]$$

where the explicit dependence of variables on time has been dropped from the notation for convenience.

Substituting all these relations back into equation (4) and dividing through by $e^{-\delta t}$ yields,

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16. See, for example, Chow (1981, Ch 18).

(4') $$0 = \max \{V(c) - \delta J + \{w[\mu_1 + \Sigma_j a_j(\mu_j - \mu_1)] - c\, J' + {}^1/_2 w^2[\sigma_1{}^2 + 2\Sigma_j a_j(\sigma_{j1}-\sigma_1{}^2) + \Sigma_j\Sigma_i a_j a_i(\sigma_{ji}-\sigma_{j1}-\sigma_{1i}+\sigma_1{}^2)]J''$$

The first order conditions for an interior maxima with respect to the decision variables c and a_j are,

(5a) $$0 = V' - J'$$

(5b) $$0 = J'(\mu_j-\mu_1) + wJ''[(\sigma_{j1}-\sigma_1{}^2) + \Sigma_i a_i(\sigma_{ji}-\sigma_{j1}-\sigma_{1i}+\sigma_1{}^2) \qquad j = 2,\ldots,S$$

To solve explicitly for the optimal c and a_j's given some instantaneous indirect utility function V(.), we need to solve these S non-dynamic implicit equations for c and the a_j's as functions of J', J" and w, then substitute these back into equation (4') and solve the resulting second-order differential equation for J(w) (subject to the initial conditions and transversality condition). Once J(.) has been obtained, explicit solutions for the optimal c and a_j's follow from equations (5a) and (5b).

Since these explicit solutions are difficult to obtain, it is usual to follow the alternative implicit procedure of defining the agent's risk tolerance as,

(6) $$\theta(w) \equiv - J'(w)/[wJ''(w)]$$

and assuming it to be <u>constant</u>. This implies that the portfolio allocation decision is independent of the saving (consumption) decision. Equation (5b) then becomes,

$$\Sigma_i a_i(\sigma_{ji}-\sigma_{j1}-\sigma_{1i}+\sigma_1{}^2) = -(\sigma_{j1}-\sigma_1{}^2) + \theta(\mu_j-\mu_1) \qquad j = 2,\ldots,S$$

These may be stacked for the (S-1) asset shares to give,

$$\Omega\underline{a} = -\underline{\nu}_1 + \theta(\underline{\mu} - \underline{\mu}_1)$$

where $\Omega_{ij} \equiv \text{cov}_t[dp_i/p_i-dp_1/p_1),(dp_j/p_j-dp_1/p_1)]$ is the ij^{th} element of the (S-1)x(S-1) covariance matrix of returns on the (S-1) zero wealth portfolios which are long in one of the j=2,...S assets and short in the first asset; $\underline{\nu}_{1j} \equiv \text{cov}_t[dp_1/p_1,(dp_j/p_j-dp_1/p_1)]$ is the j^{th} element of the vector of covariances between the returns on these portfolios and the return on the first asset; $\underline{a}$ is the vector of asset shares; $\underline{\mu}$ is the vector of mean returns on the j=2,...S and $\underline{\mu}_1$ is a (S-1) vector with every element being the mean return on the first asset.

Provided the S assets are distinct, the optimal asset shares are,

$$(7) \qquad \underline{a} = -\Omega^{-1}\underline{\upsilon}_1 + \theta\Omega^{-1}(\underline{\mu} - \underline{\mu}_1)$$

This is a vector of constants since all the terms on the right hand side of the equation have been assumed constant.[17]

The assumption of geometric Brownian motion for asset prices thus induces the classic portfolio separation results - i.e., since agents' asset demand equations differ only by one parameter of their utility functions (θ), they will be indifferent between choosing from the complete menu of S risky assets or holding shares in two mutual funds. However, these mutual fund theorems require additional funds for each country that is introduced into the model, since the agents in one country will not be satisfied by shares in the mutual funds that satisfy agents in another country.[18] Various portfolio decompositions have also been emphasised in the literature. They depend on the nature of the assumed relationships between the stochastic processes for the nominal asset returns, the general price levels and the nominal exchange rates. For example, in the compact notation used here, the decomposition of demands into a "minimum variance" portfolio ($\underline{a}^V = -\Omega^{-1}\underline{\upsilon}_1$ and $a^V_1 = 1-\Sigma_j a^V_j$) and a zero net worth "speculative" portfolio ($\underline{a}^S = \theta\Omega^{-1}(\underline{\mu}-\underline{\mu}_1)$ and $a^S_1 = -\Sigma_j a^S_j$) is apparent.[19] The asset demands may also be decomposed into a weighted average of the (same) minimum variance portfolio and a "logarithmic"

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17. The analysis in this paper will also apply if the parameters of the stochastic processes are (deterministic) functions of time, in which case the a_j's could vary deterministically through time.

18. This is due to the fact that the heterogeneity of purchasing power means that agents in different countries face different vectors of mean returns and covariance matrices. See Merton (1971) and Adler and Dumas (1983).

19. This decomposition is well known and emphasised in the international setting by Kouri (1977) and de Macedo (1982), among others. The minimum variance portfolio is the one that minimises the variance of real wealth, and the other one is so named because it is structured such that the S shares add to zero.

portfolio.[20] When asset returns are specified in nominal terms and there is a nominally riskless asset (such that $\sigma_1 dz_1 = -\sigma_Q dz_Q$), the $\underline{v}_1$ term may be expanded so that the minimum variance portfolio can itself be decomposed into the sum of a capital position and a zero net worth "inflation hedge" portfolio. If it is also assumed that the general price levels are of an expenditure weighted form, this capital position may be interpreted as an "expenditure share" portfolio.[21]

In additional to such analysis of the individual agent's asset demand functions, much attention has been paid to their aggregation to the market level. It is to this issue that I will turn next.

2.3 Equilibrium in Asset Markets

Assume that the elements of the vectors of asset demands are ordered such that the first s of the total S assets demanded by the home agent are those supplied by the home country. Similarly, assume that the first $\hat{s}$ (= S−s) assets demanded by the foreign agent are supplied by the foreign country. Then, given the aggregation of the budget constraints of individual agents,[22] there are only (S−1) independent equilibrium conditions for the S asset markets (Walras' Law),[23]

$$(8) \qquad p_j N_j = p_j n_j + e\hat{p}_{\hat{s}+j}\hat{n}_{\hat{s}+j} \qquad j = 2,\ldots,s$$

$$p_{s+j}\hat{N}_j = p_{s+j}n_{s+j} + e\hat{p}_j\hat{n}_j \qquad j = 1,\ldots,\hat{s}$$

$N_j(\hat{N}_j)$ is the exogenous supply of the j^{th} home (foreign) asset and e is the "real" (deflated) exchange rate between the home country and the foreign country, defined by

$$e \equiv E\hat{Q}/Q$$

20. So named because it is the portfolio demanded by an agent with a logarithmic indirect utility function, in which case $\theta = 1$. This decomposition is emphasised by Adler and Dumas (1983). The weights are $(1-\theta)$ and θ respectively.

21. See, for example, Branson and Henderson (1985).

22. That is, their stock constraints rather than their flow constraints. See the derivation of these budget constraints in the Appendix.

23. Since there are no taxes or transaction costs, it will always be assumed that the "Law of One Price" holds; e.g., $Qp_j = E\hat{Q}\hat{p}_{\hat{s}+j}$ which implies that $p_j = e\hat{p}_{\hat{s}+j}$.

where E (without a subscript t) is the nominal exchange rate (expressed as the number of units of home country currency per unit of foreign currency).[24]

These asset market equilibrium conditions may be used to develop an International Capital Asset Pricing Model (ICAPM) as in Adler and Dumas (1983). However, such models are capable of delivering restrictions on only (S-1) of the (S+1) prices (the S p_j's and e). In effect, the international models have introduced an additional market (the market for foreign exchange) and its price (the real exchange rate, e) into the domestic Capital Asset Pricing Model (CAPM). However, they have ignored the equilibrium condition that ensures that this foreign exchange market clears. Such an equilibrium condition may be obtained by imposing balance of payments equilibrium.[25]

<u>2.4 Balance of Payments Equilibrium</u>

Assume that the first m goods are produced by the home country. Then the stochastic differential equation representing balance of payments equilibrium for the home country is,

$$(9)\quad \Sigma_{j>s} d[n_j p_j Q] - \Sigma_{j>\hat{s}} d[\hat{n}_j E\hat{p}_j \hat{Q}] = [E\hat{c}_h \hat{Q} - c_f Q]dt + \{\Sigma_{j>s} n_j d[p_j Q] - \Sigma_{j>\hat{s}} \hat{n}_j d[E\hat{p}_j \hat{Q}]$$

where $c_f \equiv \Sigma_{i>m} c_i q_i$ is the rate of consumption of foreign goods chosen by the home country and $\hat{c}_h \equiv \Sigma_{i\leq m} \hat{c}_i \hat{q}_i$ is the rate of consumption of home goods

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24. If e is assumed constant, relative purchasing power parity holds. In that case, Fama and Farber (1979) have argued that there is no exchange rate risk in the model. Branson and Henderson (1985) correctly point out that such an assumption on a real return merely constrains any one of the three nominal processes (for E, Q and $\hat{Q}$) as a function of the other two. To the extent that the Finance literature mainly deals with real returns and the International Finance literature with nominal returns, such confusions may be purely semantic in nature. For the purposes of this paper, however, <u>ex-poste</u> deviations from purchasing power parity will be allowed. This does not necessarily imply that <u>ex-ante</u> deviations need exist.

25. Much of the literature assumes that at least some of the paramaters of the exchange rate process are endogenous, yet little attention has been paid to the necessary equilibrium conditions. Frankel (1982) simply uses the asset market constraints (assuming exogenous asset price parameters) and Stulz (1984) uses a money market equilibrium condition (assuming exogenous price level parameters) to solve for restrictions on the parameters of the exchange rate process. In a general equilibrium setting all of the constraints would hold simultaneously and it would not matter which was used to solve for a particular parameter. In this partial equilibrium model, however, the choice of balance of payments equilibrium is a natural one.

chosen by the foreign country.[26] To interpret this equation, note that the capital account is on the left and the current account on the right; the trade account is the first term on the right and the service account is the term in braces.

This equation gives restrictions on both the instantaneous mean of the exchange rate (the dt terms) and on its instantaneous variance/covariance properties (the σdz terms). Following the usual procedure and taking the instantaneous noise terms on asset prices as given (from the supply side of the asset markets), a solution for the instantaneous noise term in the real exchange rate process, $\sigma_e dz_e$, may be obtained. In combination with the asset market clearing conditions, the instantaneous means of the exchange rate and (S-1) of the asset prices may then be determined. In principle, this would yield an ICAPM that priced every asset in the world, bar one, as well as pricing exchange rate risk itself.

However, it may be shown that the restrictions contained in equation (9) cannot be satisfied by the assumed geometric Brownian motion process for the exchange rate. Hence, models that embody this assumption are not market clearing models of international asset demand. Indeed, given the assumption of rational expectations, they are internally inconsistent - i.e., the price functions implied by the aggregation of the decisions of individual agents and market clearing do not coincide with the ones the agents "assume" when determining their optimal consumption and portfolio allocation decisions. Before illustrating this inconsistency in some simplified models, and proving it for the general case, I will outline the intuition behind this result and the proof methodology that is used.

2.5 The Nature of the Inconsistency

Three propositions will be proved. They state that the investment opportunity set is spanned by any (S-1) of the S assets; that the exchange rate process has no idiosyncratic risk component; and that the exchange rate process can not be geometric Brownian motion. The third is in direct contradiction to the model's assumptions. The first two imply that at least one asset is a perfect substitute for some (portfolio of) other assets. This contracts the assumption of distinct assets (i.e., that there does not exist

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26. This condition is expressed in home country currency and is derived in the Appendix. The currency of denomination is irrelevant and extensions to more than one agent in a country or more than two countries are straight forward.

some portfolio of risky assets that is itself riskless) and implies that the covariance matrix of asset returns (Ω) is singular. Equation (7) then implies that asset demands are indeterminate.

The intuition behind these results can be deduced from the balance of payments equilibrium condition in equation (9). The trade account is the dimension of an instantaneous flow. Like the consumption terms in the agents' budget constraints (equation (2)), it is a "dt" term which is known with certainty. It will, therefore, be important for the determination of the instantaneous mean of the real exchange rate process. However, it will play no role in determining the variance/covariance properties of the real exchange rate. These will be determined solely by the interaction of the "dz" terms associated with variables that involves instantaneous real risk - i.e., those that are of a stock dimension. Hence these properties will be determined by the asset demand terms in the capital and service accounts. The "noise" (dz) term in the real exchange rate process will depend only on the noise terms in the processes for real asset prices.

There is thus no risk in the foreign exchange market other than asset market risk, so the real exchange rate carries no idiosyncratic risk. Since its "dz" term is a linear combination of those determining asset price movements, it will be possible to create a portfolio of risky assets which is itself riskless. Contrary to the assumptions of the model, the S assets will not be distinct and the investment opportunity space will be spanned by less than the full menu of assets. The covariance matrix, used to determine the agents' asset demands in equation (7), is then singular. Furthermore, the weights in the linear combination determining the exchange rate stochastic will generally not be constant over time. Hence, the market clearing exchange rate process cannot, in general, be geometric Brownian motion since its variance-covariance properties will not be constant over time.

To show the inconsistency one need only consider the "noise term" constraints implicit in equation (9). These constraints imply that either the coefficient on each dz_j term is identically zero at each instant in time, or that the exchange rate dz may be expressed as a function of the asset price dz's. Repeated differentiation of the coefficients on each dz term shows that they cannot be identically zero at each instant, hence the alternative must hold. Given this result, the three propositions follow. Since the proof is somewhat laborious, it will first be illustrated by two examples that involve only two assets. I will then present the case with many assets, before proving the three propositions.

3. Example: Two Real Riskless Assets

Assume that in each country there is an asset which is riskless in real terms for a domestic investor but carries exchange rate risk for a foreign investor (an indexed "short" government bond). There are no other assets, so S = 2. Depending on his country of residence an investor will face investment opportunity sets with real returns,

Home country	Foreign Country
$dp_1/p_1 = rdt$	$d\hat{p}_1/\hat{p}_1 = \hat{r}dt$
$dp_2/p_2 = (\hat{r}+\mu_e)dt + \sigma_e dz_e$	$d\hat{p}_2/\hat{p}_2 = (r-\mu_e+\sigma_e^2)dt - \sigma_e dz_e$

where a subscript of e indicates the parameters of the real exchange rate process, which has been assumed to be geometric Brownian motion. In terms of the model presented in Section 2, these equations define,

$$(10a)\qquad \mu_1 \equiv r \qquad\qquad \sigma_1 dz_1 \equiv 0$$

$$\mu_2 \equiv (\hat{r}+\mu_e) \qquad\qquad \sigma_2 dz_2 \equiv \sigma_e dz_e$$

for the home country and,

$$(10b)\qquad \hat{\mu}_1 \equiv \hat{r} \qquad\qquad \hat{\sigma}_1 d\hat{z}_1 \equiv 0$$

$$\hat{\mu}_2 \equiv (r-\mu_e+\sigma_e^2) \qquad\qquad \hat{\sigma}_2 d\hat{z}_2 \equiv -\sigma_e dz_e$$

for the foreign country. These definitions may then be used to obtain the processes for real wealth in the home country, dw, and, in the foreign country, $d\hat{w}$, by substituting into equation (2), where the solutions for the optimal asset shares a and $\hat{a}$ (i.e., the shares of wealth held in the asset supplied by the other country) are obtained from equation (7),

$$(11a)\qquad dw = w[r + a(\hat{r}+\mu_e-r) - c/w]dt + w[a\sigma_e dz_e]$$

$$(11b)\qquad d\hat{w} = \hat{w}[\hat{r} + \hat{a}(r-\mu_e+\sigma_e^2-\hat{r}) - \hat{c}/\hat{w}]dt + \hat{w}[-\hat{a}\sigma_e dz_e]$$

Equilibrium in the asset markets is defined by equation (8). In this version of the model the market clearing condition for the foreign asset becomes,

$$(12)\qquad p_2\hat{N}_1 = aw + (1-\hat{a})e\hat{w}$$

where the definitions of a and $\hat{a}$ given in equation (3) have been used to substitute out for the n_j's and $\hat{n}_j$'s.[27]

The balance of payments equilibrium condition, equation (9), may be rewritten as,

$$(13) \qquad ad[Qw] - \hat{a}d[Qe\hat{w}] = (e\hat{c}_h Q - c_f Q)dt + \{aQwd[Qp_2]/Qp_2 - \hat{a}Qe\hat{w}d[Qp_1]/Qp_1$$

by noting the definition of the real exchange rate and the constancy of a and $\hat{a}$ under the assumptions of the model. For this equation to hold both the drift (or dt) terms and the noise (or dz) terms on each side of the equation must be equal.

Consider the noise terms,[28]

$$(14) \qquad 0 = Qe\hat{w}\hat{a}(1-\hat{a})\sigma_e dz_e - Qwaa\sigma_e dz_e + Qwa\sigma_e dz_e$$

which can only be true if $\sigma_e dz_e$ is identically zero, or if its coefficients sum to zero. The second possibility requires that,

$$(15) \qquad 0 = \hat{a}(1-\hat{a})e\hat{w} + (1-a)aw$$

must hold for all $t\epsilon[0,\infty)$. Since equation (15) must hold at each instance, it may be differentiated by applying Ito's Lemma to both sides. Given that a and $\hat{a}$ are constant, the coefficients on the $\sigma_e dz_e$ terms in the resulting weighted sum of $de\hat{w}$ and dw, must be zero. That is,

$$0 = \hat{a}(1-\hat{a})^2 e\hat{w} + (1-a)a^2 w$$

This can be differentiated <u>ad infinitum</u> to give,

$$(16) \qquad 0 = \hat{a}(1-\hat{a})^i e\hat{w} + (1-a)a^i w \qquad i\geq 1$$

---

27. The redundant equilibrium condition is, $p_1N_1 = (1-a)w + \hat{a}e\hat{w}$, for the home country asset.

28. The coefficients on the $\sigma_Q dz_Q$ terms in equation (13) cancel out. A more direct proof by substitution is available in this simple case. Fhe method used above foreshadows the proof for the general case.

Now $\hat{a}$ is the foreign country's total holdings (in terms of budget shares) of home country assets and a is the home country's total holdings of foreign assets. There are thus only two possibilities under which equation (16) can hold. Either the price processes are such that each country holds all of its wealth in a single asset. This requires that the real exchange rate has zero variance - otherwise the asset demand equations (equation (7)) imply that agents will diversify their portfolios. The second is that $a = (1-\hat{a})$ in which case the following equation holds,

$$0 = \hat{a}e\hat{w} + (1-a)w$$

However, given the asset market equilibrium condition in equation (12), this condition will hold if and only if the total supply of the home country asset is zero. Clearly, this cannot apply in general. Hence, neither does the constraint in equation (15).

Therefore, the only solution to (the stochastic part of) the balance of payments constraint (equation (14)) is that $\sigma_e dz_e$ (real exchange rate risk) is identically zero. This means that only one of the two assets is required to span the investment opportunity space as it is perceived by either agent (the assets are perfect substitutes). The covariance matrix Ω in equation (7) is, therefore, singular and the asset demands are indeterminate. Thus the original maximisation problem is misspecified and the equilibrium real exchange rate cannot be represented by a geometric Brownian motion process as was originally assumed.

4. Example: Two Risky Assets

Consider now a version of the model that allows for risky "own" assets. Assume that each country supplies an asset which is risky in real terms for a domestic investor and also carries exchange rate risk for a foreign investor. There are no other assets and S = 2 again. The investment opportunity sets are,

Home Country

$$dp_1/p_1 = rdt + \sigma_1 dz_1$$

$$dp_2/dp_2 = (\hat{r}+\mu_e+\hat{\sigma}_{1e})dt + \sigma_e dz_e + \hat{\sigma}_1 d\hat{z}_1$$

Foreign Country

$$d\hat{p}_1/\hat{p}_1 = \hat{r}dt + \hat{\sigma}_1 d\hat{z}_1$$

$$d\hat{p}_2/\hat{p}_2 = (r-\mu_e+\sigma_e^2-\sigma_{1e})dt - \sigma_e dz_e + \sigma_1 dz_1$$

These equations define,

$$\text{(17a)}\quad \mu_1 \equiv r \qquad\qquad \sigma_1 dz_1 \equiv \sigma_1 dz_1$$

$$\mu_2 \equiv (\hat{r}+\mu_e+\hat{\sigma}_{1e}) \qquad\qquad \sigma_2 dz_2 \equiv \sigma_e dz_e + \hat{\sigma}_1 d\hat{z}_1$$

for the home country and,

$$\text{(17b)}\quad \hat{\mu}_1 \equiv \hat{r} \qquad\qquad \hat{\sigma}\, d\hat{z} \equiv \hat{\sigma}_1 d\hat{z}_1$$

$$\hat{\mu}_2 \equiv (r-\mu_e+\sigma_e^2-\sigma_{1e}) \qquad\qquad \hat{\sigma}_2 d\hat{z}_2 \equiv -\sigma_e dz_e + \sigma_1 dz_1$$

for the foreign country. These definitions may again be used to obtain the optimal asset shares a and $\hat{a}$ from the model presented in Section 2 and,

$$\text{(18a)}\quad dw = w[r + a(\hat{r}+\mu_e+\hat{\sigma}_{1e}-r) - c/w]dt + w[\sigma_1 dz_1 + a(\sigma_e dz_e + \hat{\sigma}_1 d\hat{z}_1 - \sigma_1 dz_1)]$$

$$\text{(18b)}\quad d\hat{w} = \hat{w}[\hat{r} + \hat{a}(r+\mu_e+\sigma_e^2-\sigma_{1e}-\hat{r}) - \hat{c}/\hat{w}]dt + \hat{w}[\hat{\sigma}_1 d\hat{z}_1 - \hat{a}(\sigma_e dz_e - \sigma_1 dz_1 + \hat{\sigma}_1 d\hat{z}_1)]$$

Equilibrium in the asset markets is again defined by equation (12), and the balance of payments equilibrium condition by equation (13). The constraint on the noise terms is now that either,

$$\text{(19)}\quad 0 = \hat{a}(1-\hat{a})e\hat{w} + (1-a)aw \qquad\qquad t\varepsilon[0,\infty)$$

or

$$\text{(20)}\quad 0 = \hat{\sigma}_1 d\hat{z}_1 + \sigma_e dz_e - \sigma_1 dz_1$$

The first possibility is again ruled out by the argument at the end of the previous section. The second implies that there is a riskless return on a zero wealth portfolio that is long in the foreign asset and short in the home country asset. The absence of opportunities for riskless arbitrage in equilibrium implies that these two assets generate the same expected return. They are thus perfect substitutes. Hence, the covariance matrix of returns (Ω in equation (7)) as it is perceived by either agent is singular, the original maximisation problem is misspecified, and the equilibrium real exchange rate cannot be represented by a geometric Brownian motion process with an idiosyncratic risk component.

These two examples cover all the possibilities in a world where each country supplies only one asset and S = 2. In particular, the second contains the case of a nominally riskless asset in each country, discussed by Branson and Henderson (1985). It may be obtained by interpreting $\sigma_1 dz_1$ as $-\sigma_Q dz_Q$ and $\hat{\sigma}_1 d\hat{z}_1$ as $-\hat{\sigma}_Q d\hat{z}_Q$. Equation (20) then implies that the nominal exchange rate ($E=eQ/\hat{Q}$) has zero variance, i.e., $\sigma_E dz_E = 0$. The two assets are both perceived by all agents to be nominally riskless in the agent's own domestic currency, and are thus perfect substitutes (in equilibrium). Clearly, imposing purchasing power parity (zero real exchange rate risk) in such a situation will only restrict the parameterisation of price level risk. It can not constrain nominal exchange rate risk.

5. Example: Many Assets

The above results will now be generalised to show that the equilibrium solution for (the stochastic part of) the exchange rate process in these models continues to be a function only of the stochastic parts of the processes for asset prices and therefore that the model is internally inconsistent. Assume that in addition to a real riskless asset in each country, there are (S-2) other assets of which (s-1) are supplied by the home country and ($\hat{s}$-1) are supplied by the foreign country. Depending on his nationality, an investor will face the investment opportunity sets,

Home Country

$$dp_1/p_1 = rdt$$

$$dp_j/p_j = \mu_j dt + \sigma_j dz_j \qquad j=2,\dots,s$$

$$dp_{s+1}/p_{s+1} = (\hat{r}+\mu_e)dt + \sigma_e dz_e$$

$$dp_{s+j}/p_{s+j} = (\hat{\mu}_j+\mu_e+\hat{\sigma}_{je})dt + \sigma_e dz_e + \hat{\sigma}_j d\hat{z}_j \qquad j=2,\dots,\hat{s}$$

Foreign Country

$$d\hat{p}_1/\hat{p}_1 = \hat{r}dt$$

$$d\hat{p}_j/\hat{p}_j = \hat{\mu}_j dt + \hat{\sigma}_j d\hat{z}_j \qquad j=2,\dots,\hat{s}$$

$$d\hat{p}_{\hat{s}+1}/\hat{p}_{\hat{s}+1} = (r-\mu_e+\sigma_e^2)dt - \sigma_e dz_e$$

$$d\hat{p}_{\hat{s}+j}/\hat{p}_{\hat{s}+j} = (\mu_j-\mu_e+\sigma_e^2-\sigma_{je})dt - \sigma_e dz_e + \sigma_j dz_j \qquad j=2,\dots,s$$

where $S=s+\hat{s}$. In terms of the general model of Section 2, these equations define, (21a) $\mu_1 \equiv r$ $\qquad\qquad$ $\sigma_1 dz_1 \equiv 0$

$\mu_{s+1} \equiv (\hat{r}+\mu_e)$ $\qquad\qquad$ $\sigma_{s+1} dz_{s+1} \equiv \sigma_e dz_e$

$\mu_{s+j} \equiv (\hat{\mu}_j+\mu_e+\hat{\sigma}_{je})$ $\qquad\qquad$ $\sigma_{s+j} dz_{s+j} \equiv \hat{\sigma}_j d\hat{z}_j + \sigma_e dz_e \qquad j=2,\ldots,\hat{s}$

for the home country and,

(21b) $\hat{\mu}_1 \equiv \hat{r}$ $\qquad\qquad$ $\hat{\sigma}_1 d\hat{z}_1 \equiv 0$

$\hat{\mu}_{\hat{s}+1} \equiv (r-\mu_e+\sigma_e^{\,2})$ $\qquad\qquad$ $\hat{\sigma}_{\hat{s}+1} d\hat{z}_{\hat{s}+1} \equiv -\sigma_e dz_e$

$\hat{\mu}_{\hat{s}+j} \equiv (\mu_j-\mu_e+\sigma_e^{\,2}-\sigma_{je})$ $\qquad\qquad$ $\hat{\sigma}_{\hat{s}+j} d\hat{z}_{\hat{s}+j} \equiv \sigma_j dz_j - \sigma_e dz_e \qquad j=2,\ldots,s$

for the foreign country. These definitions may then be used to obtain the optimal asset shares $\underline{a}$ and $\underline{\hat{a}}$ from equation (17) and the processes for wealth,

(22a) $dw = w[r + \Sigma a_j(\mu_j-r) + a_{s+1}(\hat{r}+\mu_e-r) + \hat{\Sigma} a_{s+j}(\hat{\mu}_j+\mu_e+\hat{\sigma}_{je}-r) - c/w]dt$

$\qquad + w[\Sigma a_j\sigma_j dz_j + a_{s+1}\sigma_e dz_e + \hat{\Sigma} a_{s+j}(\hat{\sigma}_j d\hat{z}_j+\sigma_e dz_e)]$

(22b) $d\hat{w} = \hat{w}[\hat{r} + \hat{\Sigma}\hat{a}_j(\hat{\mu}_j-\hat{r}) + \hat{a}_{\hat{s}+1}(r-\mu_e+\sigma_e^{\,2}-\hat{r}) + \Sigma\hat{a}_{\hat{s}+j}(\mu_j-\mu_e+\sigma_e^{\,2}-\sigma_{je}-\hat{r}) - \hat{c}/\hat{w}]dt$

$\qquad + \hat{w}[\hat{\Sigma}\hat{a}_j\hat{\sigma}_j d\hat{z}_j - \hat{a}_{\hat{s}+1}\sigma_e dz_e + \Sigma\hat{a}_{\hat{s}+j}(\sigma_j dz_j-\sigma_e dz_e)]$

where Σ denotes the sum over $j=2,\ldots s$ and $\hat{\Sigma}$ denotes the sum over $j=2,\ldots\hat{s}$.

Equilibrium in the asset markets is defined by equation (8) above. In this version of the model these market clearing conditions become,

(23) $p_j N_j = a_j w + \hat{a}_{\hat{s}+j} e\hat{w} \qquad j=2,\ldots s$

$p_{s+1}\hat{N}_1 = a_{s+1} w + (1-\hat{\Sigma}\hat{a}_j-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})e\hat{w}$

$p_{s+j}\hat{N}_j = a_{s+j} w + \hat{a}_j e\hat{w} \qquad j=2,\ldots,\hat{s}$

The balance of payments equilibrium condition, equation (9), may be rewritten as,

(24) $(a_{s+1}+\hat{\Sigma} a_{s+j})dQw - (\hat{a}_{\hat{s}+1}+\Sigma\hat{a}_{\hat{s}+j})dQe\hat{w} = (e\hat{c}_h Q - c_f Q)dt + \{a_{s+1}Qwd[Qp_{s+1}]/Qp_{s+1}$

$\qquad + \hat{\Sigma} a_{s+j}Qwd[Qp_{s+j}]/Qp_{s+j} - \hat{a}_{\hat{s}+1}Qe\hat{w}d[Qp_1]/Qp_1 - \Sigma\hat{a}_{\hat{s}+j}Qe\hat{w}d[Qp_j]/Qp_j$

As previously, for this equation to hold, both the drift (or dt) terms and the noise (or dz) terms on each side of the equation must be equal. Consider the noise terms,[29]

$$(25)\quad 0 = Qe\hat{w}(\hat{a}_{\hat{s}+1}+ \Sigma\hat{a}_{\hat{s}+j})(\hat{\Sigma}\hat{a}_j\hat{\sigma}_j d\hat{z}_j) + Qe\hat{w}(\hat{a}_{\hat{s}+1}+\Sigma\hat{a}_{\hat{s}+j})(1-\hat{a}_{\hat{s}+1})\sigma_e dz_e$$

$$+ Qe\hat{w}(\hat{a}_{\hat{s}+1}+ \Sigma\hat{a}_{\hat{s}+j})[\Sigma\hat{a}_{\hat{s}+j}(\sigma_j dz_j-\sigma_e dz_e)] - Qe\hat{w}(\Sigma\hat{a}_{\hat{s}+j}\sigma_j dz_j)$$

$$- Qw(a_{s+1}+\hat{\Sigma}a_{s+j})(\Sigma a_j\sigma_j dz_j) - Qw(a_{s+1}+\hat{\Sigma}a_{s+j})a_{s+1}\sigma_e dz_e$$

$$- Qw(a_{s+1}+\hat{\Sigma}a_{s+j})[\hat{\Sigma}a_{s+j}(\hat{\sigma}_j d\hat{z}_j+\sigma_e dz_e)]$$

$$+ Qw(a_{s+1}\sigma_e dz_e) + Qw[\hat{\Sigma}a_{s+j}(\hat{\sigma}_j d\hat{z}_j+ \sigma_e dz_e)]$$

which can only be true if one of the σdz's is a linear combination of the others, or if the coefficients on each σdz term sum to zero. The second possibility requires that,

$$(26a)\quad 0 = (1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})\hat{a}_{\hat{s}+i}e\hat{w} + (a_{s+1}+\hat{\Sigma}a_{s+j})a_i w \qquad i=2,\ldots,s$$

for the coefficients of $-\sigma_i dz_i$,

$$(26b)\quad 0 = (\hat{a}_{\hat{s}+1}+\Sigma\hat{a}_{\hat{s}+j})(1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})e\hat{w} + (1-a_{s+1}-\hat{\Sigma}a_{s+j})(a_{s+1}+\hat{\Sigma}a_{s+j})w$$

for the coefficients of $\sigma_e dz_e$ and,

$$(26c)\quad 0 = (\hat{a}_{\hat{s}+1}+\Sigma\hat{a}_{\hat{s}+j})\hat{a}_i e\hat{w} + (1-a_{s+1}-\hat{\Sigma}a_{s+j})a_{s+i}w \qquad i=2,\ldots,\hat{s}$$

for the coefficients of $\hat{\sigma}_i d\hat{z}_i$

These constraints must hold for all $t\epsilon[0,\infty)$. Hence Ito's Lemma may be applied to differentiate both sides of each of these relationships. Since the a_j's and $\hat{a}_j$'s are constant, this again means that the coefficients on each of the σdz in the resulting weighted sums of $de\hat{w}$ and dw must be zero. Differentiating equation (26a) gives,

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29. The coefficients on the $\sigma_Q dz_Q$ terms in equation (24) cancel out.

$$0 = (1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})\hat{a}_{\hat{s}+i}\hat{a}_{\hat{s}+k}e\hat{w} + (a_{s+i}+\hat{\Sigma}a_{s+j})a_i a_k w$$

$i=2,\ldots,s$ and $k=2,\ldots,s$

$$0 = (1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})\hat{a}_{\hat{s}+i}(1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})e\hat{w} + (a_{s+1}+\hat{\Sigma}a_{s+j})a_i(a_{s+1}+\hat{\Sigma}a_{s+j})w$$

$i=2,\ldots,s$

$$0 = (1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})\hat{a}_{\hat{s}+i}\hat{a}_k e\hat{w} + (a_{s+1}+\hat{\Sigma}a_{s+j})a_i a_{s+k}w$$

$i=2,\ldots,s$ and $k=2,\ldots,\hat{s}$

Similarly, equation (26b) requires that

$$0 = (\hat{a}_{\hat{s}+1}+\Sigma\hat{a}_{\hat{s}+j})(1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})\hat{a}_{\hat{s}+k}e\hat{w} + (1-a_{s+1}-\hat{\Sigma}a_{s+j})(a_{s+1}+\hat{\Sigma}a_{s+j})a_k w$$

$k=2,\ldots,s$

$$0 = (\hat{a}_{\hat{s}+1}+\Sigma\hat{a}_{\hat{s}+j})(1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})(1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})e\hat{w}$$
$$+ (1-a_{s+1}-\hat{\Sigma}a_{s+j})(a_{s+1}+\hat{\Sigma}a_{s+j})(a_{s+1}+\hat{\Sigma}a_{s+j})w$$

$$0 = (\hat{a}_{\hat{s}+1}+\Sigma\hat{a}_{\hat{s}+j})(1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})\hat{a}_k e\hat{w} + (1-a_{s+1}-\hat{\Sigma}a_{s+j})(a_{s+1}+\hat{\Sigma}a_{s+j})a_{s+k}w$$

$k=2,\ldots,\hat{s}$

and differentiating equation (26c) gives,

$$0 = (\hat{a}_{\hat{s}+1} + \Sigma\hat{a}_{\hat{s}+j})\hat{a}_i\hat{a}_{\hat{s}+k}e\hat{w} + (1-a_{s+1}-\hat{\Sigma}a_{s+j})a_{s+i}a_k w$$

$i=2,\ldots,\hat{s}$ and $k=2,\ldots,s$

$$0 = (\hat{a}_{\hat{s}+1}+\Sigma\hat{a}_{\hat{s}+j})\hat{a}_i(1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})e\hat{w} + (1-a_{s+1}-\hat{\Sigma}a_{s+j})a_{s+i}(a_{s+1}+\hat{\Sigma}a_{s+j})w$$

$i=2,\ldots,\hat{s}$

$$0 = (\hat{a}_{\hat{s}+1}+\Sigma\hat{a}_{\hat{s}+j})\hat{a}_i\hat{a}_k e\hat{w} + (1-a_{s+1}-\hat{S}a_{s+j})a_{s+i}a_{s+k}w$$

$i=2,\ldots,\hat{s}$ and $k=2,\ldots,\hat{s}$

This process of differentiation and equating coefficients on the different σdz terms may again be repeated ad infinitum. Notice that the term $(\hat{a}_{\hat{s}+1}+\Sigma\hat{a}_{\hat{s}+j})$ is the foreign country's total holdings (in terms of budget shares) of home country assets and the term $(a_{s+1}+\hat{\Sigma}a_{s+j})$ is the home country's total holdings of foreign assets. There are thus only three possibilities. Either the price processes are such that each country holds none of the others' assets, ever (that is, the "own" assets completely dominate the "other" assets), or each country holds all of its wealth in a single asset. Neither of these is an admissable solution to a model that is intended to explain international portfolio holdings. The third possibility is that at least one of the following equations must hold,

$$0 = (\hat{a}_{\hat{s}+1}+\Sigma\hat{a}_{\hat{s}+j})e\hat{w} + (1-a_{s+1}-\hat{\Sigma}a_{s+j})w$$

$$0 = (1-\hat{a}_{\hat{s}+1}-\Sigma\hat{a}_{\hat{s}+j})e\hat{w} + (a_{s+1}+\hat{\Sigma}a_{s+j})w$$

However, given the asset market equilibrium conditions in equation (23), the first of these conditions will hold if and only if the total supply of home country assets is zero, and the second requires that total supply of foreign country assets to be zero. Clearly, neither of these can apply. Hence, neither do the constraints in equation (26).

Therefore, the only solution to the stochastic part of the balance of payments constraint (equation (25)) is that one of the σdz terms is a linear combination of the others – i.e., that the real exchange rate dz may be expressed as a function of the asset price dz's. This implies that the investment opportunity set, as it is perceived by either agent, is spanned by (S-1) of the S assets and that the covariance matrix of excess returns, Ω in equation (7), is singular. Hence the S assets are not distinct and the original maximisation problem is misspecified.

6. The General Case

The last example assumed the existence of a real riskiness asset in each country. When there are no such assets, the analysis must be expanded to include consideration of terms in $\sigma_1 dz_1$ and $\hat{\sigma}d\hat{z}_1$. It may then be shown that the coefficient on the $\sigma_1 dz_1$ term in the balance of payments constraint is simply the sum of those on $-\sigma_i dz_i$ for $i=2,\ldots,\hat{s}$ (equation (26a)) less the one on $\sigma_e dz_e$ (equation (26b)). Similarly, the coefficient on the $\hat{\sigma}_1 d\hat{z}_1$ term is that on $\sigma_e dz_e$ less the sum of those on $\hat{\sigma}_i d\hat{z}_i$ for $i=2,\ldots,s$ (equation (26c).

Hence the additional restrictions, that the coefficients on $\sigma_1 dz_1$ and $\hat{\sigma}_1 d\hat{z}_1$ are also zero, are not independent of those in equation (26). The above analysis then implies that the only solution to the stochastic part of the balance of payments constraints is that one of the σdz's is a linear combination of the others such that,

$$(27) \qquad 0 = \Sigma_i \pi_i (\sigma_i dz_i - \sigma_1 dz_1) + (\pi_0 - \hat{\Sigma}_i \hat{\pi}_i)(\hat{\sigma}_1 d\hat{z}_1 + \sigma_e dz_e - \sigma_1 dz_1)$$

$$+ \hat{\Sigma}_i \hat{\pi}_i (\hat{\sigma}_i d\hat{z}_i + \sigma_e dz_e - \sigma_1 dz_1)$$

where,

$$\pi_0 \equiv (\hat{a}_{\hat{s}+1} + \Sigma \hat{a}_{\hat{s}+j})(1 - \hat{a}_{\hat{s}+1} - \Sigma \hat{a}_{\hat{s}+j}) e\hat{w} + (1 - a_{s+1} - \hat{\Sigma} a_{s+j})(a_{s+1} + \hat{\Sigma} a_{s+j}) w$$

$$\pi_i \equiv -(1 - \hat{a}_{\hat{s}+1} - \Sigma \hat{a}_{\hat{s}+j}) \hat{a}_{\hat{s}+i} e\hat{w} - (a_{s+1} + \hat{\Sigma} a_{s+j}) a_i w \qquad i=2,\dots,s$$

$$\hat{\pi}_i \equiv (\hat{a}_{\hat{s}+1} + \Sigma \hat{a}_{\hat{s}+j}) \hat{a}_i e\hat{w} + (1 - a_{s+1} - \hat{\Sigma} a_{s+j}) a_{s+i} w \qquad i=2,\dots,\hat{s}$$

Equation (27) shows that a linear combination of the returns on the zero wealth portfolios constructed by taking a long position in one of the $j=2,\dots,S$ assets and a short position in the home country's first asset, is once again riskless. Contrary to the assumption of distinct assets, riskless portfolios of risky assets can be created. This implies that the investment opportunity set, as it is perceived by the home agent, is spanned by (S-1) of the S assets and that his covariance matrix of excess returns, Ω in equation (7), is singular. A simple rearrangement of terms in equation (27) shows that,

$$(27') \qquad 0 = \hat{\Sigma}_i \hat{\pi}_i (\hat{\sigma}_i d\hat{z}_i - \hat{\sigma}_1 d\hat{z}_1) - (\pi_0 + \Sigma_i \pi_i)(\sigma_1 dz_1 - \sigma_e dz_e - \hat{\sigma}_1 d\hat{z}_1)$$

$$+ \Sigma_i \pi_i (\sigma_i dz_i - \sigma_e dz_e - \hat{\sigma}_1 d\hat{z}_1)$$

which implies that the investment opportunity set perceived by the foreign agent has an identical problem. Hence, the S assets are not distinct and the original maximisation problem is misspecified. From this property of the most general case, the following propositions may be proved.

<u>Proposition 1</u>: Under the assumption of market clearing prices, the investment opportunity set perceived by any agent is spanned by any S-1 of the S available assets.

Proposition 2: The market clearing exchange rate process cannot have an idiosyncratic risk (dz) component. Rather, exchange rate risk is a linear combination of asset market risks.

These propositions rest on the result in equation (27) which gives $\sigma_e dz_e$ as a linear combination of the stochastic parts of the asset prices. In order for the exchange rate to be a geometric Brownian motion process at all, the weights in this linear combination (the π's) must be constant. To see that even this possibility is inconsistent with the model, consider the case where each country supplies a real riskless asset ($\sigma_1 dz_1$ and $\hat{\sigma}_1 d\hat{z}_1$ are zero) and a single risky asset (s and $\hat{s}$ are both two). Then,

(28) $$\sigma_e dz_e = -(\pi_2/\pi_0)\sigma_2 dz_2 - (\hat{\pi}_2/\hat{\pi}_0)\hat{\sigma}_2 d\hat{z}_2$$

where,

$$\pi_0 \equiv (\hat{a}_3+\hat{a}_4)(1-\hat{a}_3-\hat{a}_4)e\hat{w} + (1-a_3-a_4)(a_3+a_4)w$$

$$\pi_2 \equiv -(1-\hat{a}_3-\hat{a}_4)\hat{a}_4 e\hat{w} - (a_3+a_4)a_2 w$$

$$\hat{\pi}_2 \equiv (\hat{a}_3+\hat{a}_4)\hat{a}_2 e\hat{w} + (1-a_3-a_4)a_4 w$$

a_2 is the share of wealth invested in one's own country's risky asset, and a_3 and a_4 are the shares invested in the other country's riskless and risky assets respectively. Since all the a_j's are constant, the weights will be constant if and only if the ratio of home country to foreign country wealth $(w/e\hat{w})$ is constant over time. Ito's Lemma and the budget constraints in equation (22) may be used to show that this requires,

(29) $$-[(a_3+a_4)w - (1-\hat{a}_3-\hat{a}_4)e\hat{w}]\sigma_e dz_e = [a_2 w - \hat{a}_4 e\hat{w}]\sigma_2 dz_2 + [a_4 w - \hat{a}_2 e\hat{w}]\hat{\sigma}_2 d\hat{z}_2]$$

which cannot hold simultaneously with equation (28). Hence the weights in equation (28) cannot be constants. This suggests a third proposition, which may be proved for the more general case.

Proposition 3: The market clearing exchange rate process cannot be geometric Brownian motion.

7. Concluding Observations

This paper has established that the model conventionally used in the literature on international portfolio analysis is internally inconsistent. Contrary to the assumptions of this model, the stochastic part of the market clearing exchange rate is a linear combination (with non-constant weights) of the risk components of the asset price processes. It cannot have an idiosyncratic risk component, nor can it be a geometric Brownian motion process.

The reason for this is clear. Examination of the budget constraint reveals that consumption is an instantaneous flow and asset holdings are instantaneous stocks. Instantaneous uncertainty only enters an agent's decisions via stock variables - i.e., via the real capital gains/losses he will accrue over time due to his asset holdings. Hence an agent's flow demand for foreign exchange has a risk component that is a function of asset market risk only. There is no uncertainty with respect to his demand for foreign exchange to make foreign consumption purchases. Therefore, the stochastic part of the net demand for foreign exchange by each country is simply a function only of the real risk in asset markets. The equilibrium real exchange rate stochastics can only be a function of these risks. This leads to the inconsistency result.

The inconsistency is important because one of the assumptions of the model is that agents have rational expectations. A fundamental implication of this assumption is that the price functions and dynamics "assumed" by agents in determining their portfolio demands are indeed the ones obtained from the interaction of the (assumed) supply behaviour and the aggregation of the optimal demand decisions of all agents. The model violates this necessary condition. Therefore it does not provide a micro-foundations theory of international asset demand.

This problem only arises in a model with market clearing exchange rates. Perhaps one could claim that while asset markets clear, foreign exchange markets do not? In the context of this model, however, it cannot be assumed that exchange rates are fixed, or "crawling". In that case, the assumption of smooth continuous processes for exchange rates would be inappropriate. Even to think of exchange rate determination as a managed float, taking the parameters of its stochastic process as exogenous, is inconsistent with the spirit of the model and much of the literature in which it is imbedded. The discussion in Branson and Henderson (1985), for example, assumes that the exchange rate is endogenous and Frankel (1982) uses asset market equilibrium constraints to solve for the instantaneous means of exchange rate process, rather than the means of asset price processes.

Moreover, given the continuous time structure of the model, consumption is of an instantaneous flow dimension that is known with certainty. Only portfolio holdings are of a stock dimension that involves instantaneous real risk. The lack of clearing in goods markets (or markets other than asset markets) will not, therefore, logically impinge on the demonstration of inconsistency presented above.

The analysis involves a number of other assumptions that could be relaxed without removing the inconsistency. Each of these will be treated in turn:

a) Two agents and two countries. At the cost of severely complicating the algebra, the above argument may be reconstructed for more than one agent in each country or more than two countries. This merely adds additional terms to the constraints in the expression for the noise terms of the balance of payments. The process of differentiating and collecting terms still generates the same inconsistency result and the same three propositions.

b) Constant risk tolerance. Allowing the coefficients of risk tolerance $\theta(w)$ and $\hat{\theta}(\hat{w})$to vary stochastically means that the asset shares are not constant and equation (24) does not represent balance of payments equilibrium. However, examination of the correct equilibrium condition in equation (9) shows that the same arguments apply. The additional risk introduced to agents' foreign exchange demands is the risk associated with movements in real wealth, which is only asset market risk. There is still no reason for idiosyncratic risk in the foreign exchange market, and the three propositions continue to hold.

c) Homothetic utility functions. Relaxing this assumption, of itself, only adds terms reflecting hedges against unfavourable shifts in consumption prices to agents' asset demand equations. It does not change the essence of the results. However, if the coefficient of risk tolerance is variable as well, there will be a channel for non-asset market risk to enter the foreign exchange market. This is because the θ's are then functions of both real wealth and the vector of consumption prices.[30] To that extent, the noise term in the exchange rate process is a linear combination of those in the processes for both asset prices and consumption prices; all of the S assets are required to span the investment opportunity set, and the covariance matrix is invertible. Propositions 1 and 2 are then invalidated. Nonetheless, it is unlikely that a specification of utility functions can be found that also allows the weights to be constant. The equilibrium exchange rate is thus not geometric Brownian motion, so Proposition 3 continues to hold. Moreover, Adler and Dumas (1983) conclude that for many countries the covariance between exchange rates and consumer price indexes is low (in monthly data). This suggests that goods market risk is not a large component of exchange rate risk. Hence, the homotheticity assumption may not be an inappropriate modeling strategy.

d) Capital gains are sole source of income. The addition of wage income or transfer payments to agents' budget constraints does not change the results of the analysis unless that income is of a stock dimension (i.e., involves instantaneous uncertainty). Even with such a source of additional uncertainty, it is not clear that conditions exist under which the weights (in the solutions for the parameters of the exchange rate process) are constant. Thus Proposition 3 is likely to remain.

e) No monetary holdings in the model. Introducing model holdings via agents' instantaneous utility functions (directly as in Kouri (1977) or indirectly through a production function for consumption services as in Stulz (1984)) changes the asset demand equations but does not provide an independent source of risk in the foreign exchange market. All the propositions remain valid unless agents also have an associated uncertain (i.e., stock dimension) transfer payment from the government injected into their budget constraints.

f) No government demand for foreign exchange. A micro-foundations model that allows for (continuous, stochastic) government intervention in the foreign exchange market could be constructed, but then one would want to explicitly specify (or derive) the governments' intervention rules. It is not sufficient to assume that the intervention policy is whatever is required to produce a geometric Brownian motion process for the exchange rate.

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30. Actually the vector of consumption prices can be replaced by two price indices - one based on average and the other on marginal expenditure shares. See Breeden (1979) and Stulz (1981).

These arguments suggest that relaxing any (combination) of these assumptions does not produce an equilibrium exchange rate process that is consistent with the assumptions of the model. A condition for rational expectations equilibrium is that the price functions assumed by agents are the same as those implied by the aggregation of their consumption and portfolio allocation decisions. Therefore, this model (which is commonly encountered in the literature) does not provide a consistent micro-foundations theory of international asset demand.

The model may be made into a consistent theory if more complicated Ito processes for asset prices and the exchange rate (where the parameters of the distributions are themselves stochastic) are allowed. Such models have been utilised in the international literature but, as yet, the constraints imposed by equilibrium in foreign exchange markets have not been explored. Given the assumption of market clearing under rational expectations, it may be fruitful to extend these models to explicitly allow agents to take into account the result that the equilibrium exchange rate process is a linear combination of the other stochastic processes.

APPENDIX

DERIVATION OF CONTINUOUS TIME CONSTRAINTS

The assumption of Ito processes for the dynamics of asset prices and exchange rates means that functions involving these variables will be right-continuous functions of time. They will be differentiable in the stochastic, but not in the normal, sense. Thus, budget constraints and equilibrium conditions need to be specified consistently with the properties of stochastic calculus. This Appendix presents consistent derivations of these constraints from an underlying discrete time model.

A.1 Budget Constraints

Assume there are planning periods of h time units in length and that agents make expenditure/investment plans at time t for the period [t,t+h) such that equilibrium occurs at the beginning of the period, i.e., at t. So, given their stock of real wealth, $w(t)$, and current (deflated) prices, $q_i(t)$ and $p_j(t)$, agents choose their instantaneous rates of flow of consumption goods for the period and their asset stocks to be held during the period,

$$(A.1) \qquad w(t) = \Sigma_j n_j(t)p_j(t) + \Sigma_i c_i(t)q_i(t)h$$

At the end of this planning period, the agent's stock of wealth will be increased (decreased) by the amount of capital gains (losses),

$$w(t+h) = \Sigma_j n_j(t)p_j(t) + \Sigma_j n_j(t)[p_j(t+h)-p_j(t)]$$

that is,

$$(A.2) \qquad w(t+h) = \Sigma_j n_j(t)p_j(t+h)$$

This means that wealth at the end of period [t-h,t), i.e., just before the beginning of period [t,t+h), is

$$(A.2') \qquad w(t) = \Sigma_j n_j(t-h)p_j(t)$$

The budget constraint for period [t,t+h) is given by equating equations (A.1) and (A.2'). However, from Foley (1975), May (1970) and Meyer (1975) one would expect that this single budget constraint in discrete time would give rise to two constraints in continuous time as the planning interval is allowed to shrink to zero. The first of these, the stock (or balance sheet) constraint, may be derived from (A.2) by using the right-continuity property of Ito processes and taking limits as $h \to 0$,

$$\text{(A.3)} \qquad w(t) = \Sigma_j n_j(t) p_j(t)$$

The second is the flow (or financing) constraint. Taking equation (A.1) from equation (A.2) gives,

$$w(t+h) - w(t) = \Sigma_j n_j(t)[p_j(t+h) - p_j(t)] - \Sigma_i c_i(t) q_i(t) h$$

Again take limits as $h \to 0$ to give,

$$\text{(A.4)} \qquad dw(t) = \Sigma_j n_j(t) dp_j(t) - \Sigma_i c_i(t) q_i(t) dt$$

In order to illustrate the similarity of the structure of this constraint with the balance of payments equilibrium condition to be derived below, consider Merton's (1971) derivation. Equating equation (A.1) and equation (A.2') in the discrete time model gives,

$$-\Sigma_i c_i(t) q_i(t) h = \Sigma_j [n_j(t) - n_j(t-h)] p_j(t)$$

which needs to be incremented to take advantage of the right-continuity property,

$$-\Sigma_i c_i(t+h) q_i(t+h) h = \Sigma_j [n_j(t+h) - n_j(t)] p_j(t+h)$$

From Ito's Lemma, it is known that the difference expression on the right of this equation contains terms of order $\sqrt{h}$, and therefore the equation must be expanded before taking limits,

$$-\Sigma_i c_i(t+h) q_i(t+h) h = \Sigma_j [n_j(t+h) - n_j(t)][p_j(t+h) - p_j(t)] + \Sigma_j [n_j(t+h) - n_j(t)] p_j(t)$$

Taking limits as $h \to 0$ yields,

$$-\Sigma_i c_i(t) q_i(t) dt = \Sigma_j dn_j(t) dp_j(t) + \Sigma_j dn_j(t) p_j(t)$$

which may be compared with equation (A.4) to deduce,

(A.5) $dw(t) = \Sigma_j n_j(t)dp_j(t) + \Sigma_j dn_j(t)p_j(t) + \Sigma_j dn_j(t)dp_j(t)$

This is precisely the same as the result of applying Ito's Lemma to the stock constraint, equation (A.3). The last term does not vanish on substitution as it does in the normal calculus because it involves the product of terms of order $\sqrt{h}$ rather than of order h.

For much of the analysis it is convenient to work in terms of asset shares,

(A.6) $a_j(t) \equiv n_j(t)p_j(t)/w(t)$

Substituting into equation (A.3), the stock constraint becomes,

(A.3') $1 = \Sigma_j a_j(t)$

Using this in equation (A.4) gives the flow constraint,

(A.4') $dw(t) = \Sigma_j a_j(t)[dp_j(t)/p_j(t)]w(t) - \Sigma_i c_i(t)q_i(t)dt$

These two constraints may now be combined,

(A.7) $dw(t) = \{[1-\Sigma_j a_j(t)][dp_1(t)/p_1(t)] + \Sigma_j a_j(t)[dp_j(t)/p_j(t)]\ w(t)$

$- \{\Sigma_i c_i(t)q_i(t)\ dt$

where the summation over j now runs from 2 to S. There is no longer an explicit constraint on the a_j's because they are now only defined for j=2,...S. Hence (A.7) may be used as the sole budget constraint.

A.2 Equilibrium in Asset Markets

Without loss of generality, assume that there are just two countries. (A "hat" will be used to denote a foreign agent.) From the point of view of a home agent, the first s of the S assets available are supplied by the home country. As seen by a foreign agent, however, these are the last s of the S assets he demands. His first $\hat{s}$ (=S-s) assets are supplied by the foreign country, as are the last $\hat{s}$ assets demanded by a home agent.

Equilibrium occurs at the beginning of the period, so equate asset supply and demand to give,

$$p_j(t)N_j(t) = \Sigma_k p_j(t)n^k{}_j(t) + \Sigma_k e(t)\hat{p}_{\hat{s}+j}(t)\hat{n}^k{}_{\hat{s}+j}(t) \qquad j=1,\ldots,s$$

$$p_{s+j}(t)\hat{N}_j(t) = \Sigma_k p_{s+j}(t)n^k{}_{s+j}(t) + \Sigma_k e(t)\hat{p}_j(t)\hat{n}^k{}_j(t) \qquad j=1,\ldots\hat{s}$$

where N_j ($\hat{N}_j$) is the (exogenous) stock of the j^{th} home (foreign) asset, n^k_j ($\hat{n}^k{}_{\hat{s}+j}$) is the k^{th} home (foreign) agent's demand for the j^{th} home country asset, $p_j{}^k(\hat{p}^k{}_{\hat{s}+j})$ is the (deflated) price of the j^{th} home asset in the home (foreign) currency and e is the real (deflated) exchange rate. Taking limits as $h \to 0$,

(A.8) $$p_j(t)N_j(t) = \Sigma_k p_j(t)n^k{}_j(t) + \Sigma_k e(t)\hat{p}_{\hat{s}+j}(t)\hat{n}^k{}_{\hat{s}+j}(t) \qquad j=1,\ldots,s$$

$$p_{s+j}(t)\hat{N}_j(t) = \Sigma_k p_{s+j}(t)n^k{}_{s+j}(t) + \Sigma_k e(t)\hat{p}_j(t)\hat{n}^k{}_j(t) \qquad j=1,\ldots\hat{s}$$

which are the market clearing conditions for the asset markets. This gives S conditions, of which only (S-1) are independent given the aggregation of the individual (stock) budget constraints (Walras' Law).

A.3 Balance of Payments Equilibrium

For derivation purposes, consider the balance of payments of the home country, expressed in its own currency, when the first m goods are produced by it.

At the end of period [t-h,t) the net holdings of foreign assets by the home country are $\Sigma_k\Sigma_{j>s}n^k{}_j(t-h)p_j(t)Q(t)$, whereas at the beginning of period [t,t+h) net holdings are $\Sigma_k\Sigma_{j>s}n^k{}_j(t)p_j(t)Q(t)$. The corresponding holdings of home country assets by the foreign country are $\Sigma_k\Sigma_{j>\hat{s}}\hat{n}^k{}_j(t-h)E(t)\hat{p}_j(t)\hat{Q}(t)$ and $\Sigma_k\Sigma_{j>\hat{s}}\hat{n}^k{}_j(t)E(t)\hat{p}_j(t)\hat{Q}(t)$ respectively. The rate of nominal consumption of foreign goods chosen by the home country will be $\Sigma_k\Sigma_{i>m}c^k{}_i(t)q_i(t)Q(t)$ ($\equiv c_f(t)Q(t)$) and the rate of consumption of home goods chosen by the foreign country is $\Sigma_k\Sigma_{i\leq m}c^k{}_i(t)E(t)\hat{q}_i(t)\hat{Q}(t)$ ($\equiv E(t)\hat{c}_h(t)\hat{Q}(t)$), where the subscript i is ordered as in the home country.

Given these relationships, the balance of payments equilibrium condition for the home country is,

$$- [E(t)\hat{c}_h(t)\hat{Q}(t) - c_f(t)Q(t)]h = \Sigma_k\Sigma_{j>\hat{s}}[\hat{n}^k{}_j(t) - \hat{n}^k{}_j(t-h)]E(t)\hat{p}_j(t)\hat{Q}(t)$$

$$- \Sigma_k\Sigma_{j>s}[n^k{}_j(t) - n^k{}_j(t-h)]p_j(t)Q(t)$$

where the trade balance appears on the left and asset transactions on the right. As in Merton's derivation of the budget constraints shown above, this equation needs to be incremented to take advantage of the right-continuity property of Ito processes,

$$- [E(t+h)\hat{c}_h(t+h)\hat{Q}(t+h) - c_f(t+h)Q(t+h)]h$$

$$= \Sigma_k\Sigma_{j>\hat{s}}[\hat{n}^k{}_j(t+h) - \hat{n}^k{}_j(t)]E(t+h)\hat{p}_j(t+h)\hat{Q}(t+h)$$

$$- \Sigma_k\Sigma_{j>s}[n^k{}_j(t+h) - n^k{}_j(t)]p_j(t+h)Q(t+h)$$

Again, the difference expressions on the right hand side of the equation contain terms of order $\sqrt{h}$ and need to be expanded,

$$- [E(t+h)\hat{c}_h(t+h)\hat{Q}(t+h) - c_f(t+h)Q(t+h)]h$$

$$= \Sigma_k\Sigma_{j>\hat{s}}[\hat{n}^k{}_j(t+h) - \hat{n}^k{}_j(t)]E(t)\hat{p}_j(t)\hat{Q}(t)$$

$$+ \Sigma_k\Sigma_{j>\hat{s}}[\hat{n}^k{}_j(t+h) - \hat{n}^k{}_j(t)][E(t+h)\hat{p}_j(t+h)\hat{Q}(t+h) - E(t)\hat{p}_j(t)\hat{Q}(t)]$$

$$- \Sigma_k\Sigma_{j>s}[n^k{}_j(t+h) - n^k{}_j(t)]p_j(t)Q(t)$$

$$- \Sigma_k\Sigma_{j>s}[n^k{}_j(t+h) - n^k{}_j(t)][p_j(t+h)Q(t+h) - p_j(t)Q(t)]$$

Taking limits gives,

$$\text{(A.9)} \qquad - [E\hat{c}_h\hat{Q} - c_fQ]dt = \Sigma_k\Sigma_{j>\hat{s}}[d\hat{n}^k{}_j]E\hat{p}_j\hat{Q} + \Sigma_k\Sigma_{j>\hat{s}}[d\hat{n}^k{}_j]d[E\hat{p}_j\hat{Q}]$$

$$- \Sigma_k\Sigma_{j>s}[dn^k{}_j]p_jQ - \Sigma_k\Sigma_{j>s}[dn^k{}_j][dp_jQ]$$

which is the stochastic differential equation representing balance of payments equilibrium for the home country.

To interpret this equation, define the net foreign asset position of the home country, F(t), as

$$\text{(A.10)}\qquad F(t) \equiv \Sigma_k \Sigma_{j>s} n^k_j(t) p_j(t) Q(t) - \Sigma_k \Sigma_{j>\hat{s}} \hat{n}^k_j(t) E(t) \hat{p}_j(t) \hat{Q}(t)$$

Applying Ito's Lemma to this and using equation (A.9) gives,

$$\text{(A.11)}\qquad dF = [E\hat{c}_h\hat{Q} - c_f Q]dt + \{\Sigma_k \Sigma_{j>s} n^k_j [dp_j Q] - \Sigma_k \Sigma_{j>\hat{s}} \hat{n}^k_j d[E\hat{p}_j \hat{Q}]$$

where the capital account is on the left and the current account on the right; the trade account is the first term on the right and the service account is the term in curly brackets.

REFERENCES

Adler, M. and B. Dumas (1983). International Portfolio Choice and Corporation Finance: A Synthesis. Journal of Finance XXXVIII, 925-984.

Branson, W.H. and D.W. Henderson (1985). The Specification and Influence of Asset Markets. In R. Jones and P.B. Kenen (eds) (1985). Handbook of International Economics. Volume 2. Amsterdam: North-Holland, 749-805.

Breeden, D.T. (1979). An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities. Journal of Financial Economics VII, 265-296.

Chow, G.C. (1981). Econometric Analysis by Control Methods. New York: Wiley.

Cox, J.C., J.E. Ingersoll and S.A. Ross (1985a). An Intertemporal General Equilibrium Model of Asset Prices. Econometrica LIII, 363-384.

Cox, J.C., J.E. Ingersoll and S.A. Ross (1985b). A Theory of the Term Structure of Interest Rates. Econometrica LIII, 385-409.

Fama, E.F. and A. Farber (1979). Money, Bonds and Foreign Exchange. American Economic Review LXIX, 639-649.

Foley, D.K. (1975). On Two Specifications of Asset Equilibrium in Macroeconomic Models. Journal of Political Economy LXXXIII, 305-324.

Frankel, J.A. (1982). In Search of the Exchange-Risk Premium: A Six-Currency Test Assuming Mean-Variance Optimization. Journal of International Money and Finance I, 255-274.

Hodrick, R. (1981). International Asset Pricing with Time-Varying Risk Premia. Journal of International Economics XI, 573-577.

Kouri, P.J.K. (1977). International Investment and Interest Rate Linkages Under Flexible Exchange Rates. In R.A. Aliber (ed) (1977). The Political Economy of Monetary Reform. London: Macmillan, 74-96.

de Macedo, J.B. (1982). Portfolio Diversification Across Currencies. In R.N. Cooper, P.B. Kenen, J.B. de Macedo and J. van Ypersele (eds) (1982). The International Monetary System Under Flexible Exchange Rates: Global, Regional and National. Cambridge: Ballinger, 69-100.

May, J. (1970). Period Analysis and Continuous Analysis in Patinkin's Macroeconomic Model. Journal of Economic Theory II, 1-9.

Merton, R.C. (1969). Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case. Review of Economics and Statistics LI, 247-257.

Merton, R.C. (1971). Optimum Consumption and Portfolio Rules in a Continuous-Time Model. Journal of Economic Theory III, 373-413.

Merton, R.C. (1973). An Intertemporal Capital Asset Pricing Model. Econometrica XLI, 867-887.

Meyer, L.H. (1975). The Balance Sheet Identity, the Government Financing Constraint and the Crowding-Out Effect. Journal of Monetary Economics I, 65-78.

Rosenberg, B. and J.A. Ohlson (1976). The Stationary Distribution of Returns and Portfolio Separation in Capital Markets: a Fundamental Contradiction. Journal of Financial and Quantitative Analysis XI, 393-401.

Stulz, R.M. (1981). A Model of International Asset Pricing. Journal of Financial Economics IX, 383-406.

Stulz, R.M. (1984). Currency Preferences, Purchasing Power Risks, and the Determination of Exchange Rates in an Optimizing Model. Journal of Money, Credit and Banking XVI, 302-316.