RDP 2015-11 – Unprecedented Changes in the Terms of Trade: Online Appendix

Mariano Kulish* and Daniel M. Rees†

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*School of Economics, UNSW, m.kulish@unsw.edu.au
†Economic Research, Reserve Bank of Australia reesd@rba.gov.au

1 The Model

This section outlines the non-linear model.

1.1 Households

The representative household maximises its expected lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \zeta_t U\left(C_t, L_{H,t}, L_{N,t}, L_{X,t}\right)$$

where ζ_t is an intertemporal preference shock:

$$\ln \zeta_t = \rho_\zeta \ln \zeta_{t-1} + u_t^\zeta \tag{1}$$

The household's utility function is given by:

$$U\left(C_t, L_{H,t}, L_{N,t}, L_{X,t}\right) = \ln\left(C_t - hC_{t-1}\right) - \frac{\epsilon_t^L}{1+\nu}\left[\xi_H L_{H,t}^{1+\omega} + \xi_N L_{N,t}^{1+\omega} + \xi_X L_{X,t}^{1+\omega}\right]^{\frac{1+\nu}{1+\omega}}$$

where ϵ_t^L is a shock to labour supply:

$$\ln \epsilon_t^L = \rho_\epsilon \ln \epsilon_{t-1}^L + u_t^{\epsilon^L} \tag{2}$$

Households maximise subject to the budget constraint:

$$\begin{aligned} P_t C_t + P_{I,t} I_t + B_{t+1} + S_t B_{t+1}^* \leq \left(1 + R_{t-1}\right) B_t + \left(1 + R_{t-1}^F\right) S_t B_t^* \\ + W_{H,t} L_{H,t} + W_{N,t} L_{N,t} + W_{X,t} L_{X,t} + R_{N,t}^K K_{N,t} + R_{H,t}^K K_{H,t} + R_{X,t}^K K_{X,t} \\ + \Gamma_{N,t} + \Gamma_{H,t} + \Gamma_{X,t} - T_t \end{aligned}$$

and the capital accumulation equations:

$$K_{H,t+1} = (1-\delta) K_{H,t} + V_t \left[1 - \Upsilon\left(\frac{I_{h,t}}{I_{h,t-1}}\right)\right] J_{H,t} \tag{3}$$

$$K_{N,t+1} = (1-\delta) K_{N,t} + V_t \left[1 - \Upsilon\left(\frac{I_{n,t}}{I_{n,t-1}}\right)\right] J_{N,t} \tag{4}$$

$$K_{X,t+1} = (1-\delta) K_{X,t} + V_t \left[1 - \Upsilon\left(\frac{I_{x,t}}{I_{x,t-1}}\right)\right] J_{X,t} \tag{5}$$

where:

$$V_t = \frac{v}{\left(1 + z_I\right)^t} \tilde{V}_t \tag{6}$$

$$\ln \tilde{V}_t = \rho_V \ln \tilde{V}_{t-1} + u_t^V \tag{7}$$

Note that $J_{N,t}$ refers to investment that produces capital for use in the non-tradeable sector. It is distinct from $I_{N,t}$ which refers to non-tradeable goods used to produce investment goods via the investment goods aggregator.
The interest rate that the household receives on foreign bonds follows the process:

$$R_t^F = R_t^* \exp\left[-\psi_b \left(\frac{S_t B_t^*}{P_t Y_t} - b^*\right) + \tilde{\psi}_{b,t}\right] \tag{8}$$

where b^* is the steady-state net foreign asset-to-GDP ratio, $\tilde{\psi_{b,t}}$ is a risk-premium shock that follows the process:

$$\tilde{\psi}_{b,t} = \rho_\psi \tilde{\psi}_{b,t} + u_{\psi,t} \tag{9}$$

and R^*_t follows the process

$$\ln R^*_t = (1 - \rho_{R^*}) \ln R^* + \rho_{R^*} \ln R^*_{t-1} + u_{R^*,t} \tag{10}$$

The household's consumption bundle is a CES aggregate of traded and non-traded goods, while the traded goods is itself a CES aggregate of home- and foreign-produced traded goods:

$$C_t = \left[\gamma_{T,t}^{\frac{1}{\eta}} C_{T,t}^{\frac{\eta-1}{\eta}} + \gamma_{N,t}^{\frac{1}{\eta}} C_{N,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$
$$C_{T,t} = \frac{C_{H,t}^{\gamma_H} C_{F,t}^{\gamma_F}}{\gamma_H^{\gamma^H} \gamma_F^{\gamma_F}}$$

Note that the Cobb-Douglas specification allows us to assume that $\gamma_{H,t} = \gamma_H$ and so on.
The non-traded, home-produced traded and imported consumption goods are themselves bundles of imperfectly substitutable goods:

$$C_{j,t} \equiv \left(\int_0^1 C_{j,t}(i)^{\frac{\theta_j - 1}{\theta_j}} di \right)^{\frac{\theta_j}{\theta_j - 1}}$$

The price indices corresponding to the consumption goods aggregates are:

$$P_t = \left[\gamma_{T,t} P_{T,t}^{1-\eta} + \gamma_{N,t} P_{N,t}^{1-\eta} \right]^{\frac{1}{1-\eta}} \tag{11}$$
$$P_{T,t} = P_{H,t}^{\gamma_H} P_{F,t}^{\gamma_F} \tag{12}$$

The investment good is similarly a CES aggregate of non-traded and traded goods:

$$I_t = (1 + z_v)^t \frac{I_{T,t}^{\gamma_T^I} I_{N,t}^{\gamma_N^I}}{{\gamma_T^I}^{\gamma_T^I} {\gamma_N^I}^{\gamma_N}}$$
$$I_{T,t} = \frac{I_{H,t}^{\gamma_H^I} I_{F,t}^{\gamma_F^I}}{{\gamma_H^I}^{\gamma^H} {\gamma_F^I}^{\gamma_F}}$$

The price indices corresponding to the investment goods aggregates are:

$$P_t^I = (1 + z_v)^{-t} {P_{T,t}^I}^{\gamma_T^I} P_{N,t}^{\gamma_N^I} \tag{13}$$
$$P_{T,t}^I = P_{H,t}^{\gamma_H^I} P_{F,t}^{\gamma_F^I} \tag{14}$$

The non-traded, home-produced traded and imported investment goods are themselves bundles of imperfectly substitutable goods:

$$I_{j,t} \equiv \left(\int_0^1 I_{j,t}(i)^{\frac{\theta_j - 1}{\theta_j}} di \right)^{\frac{\theta_j}{\theta_j - 1}}$$

1.2 Production

1.2.1 Commodity Firms

Commodity firms produce using the Cobb-Douglas production function:

$$Y_{X,t} = A_t Z_{X,t} K_{X,t}^{\alpha_X} \left(Z_t L_{H,t}\right)^{1-\alpha_X} \tag{15}$$

where $Z_{X,t}$ is a stationary sector-specific TFP shock:

$$\begin{aligned} Z_{X,t} &= (1+z_X)^t \tilde{Z}_{X,t} \\ \ln\left(\tilde{Z}_{X,t}\right) &= \rho_X \ln\left(\tilde{Z}_{X,t-1}\right) + u_{X,t} \end{aligned} \tag{16}$$

A_t is a stationary technology shock that is common across sectors:

$$\ln\left(A_t\right) = \rho_a \ln\left(A_{t-1}\right) + u_{A,t} \tag{17}$$

and Z_t is a labour-augmenting technology shock that is common across sectors whose growth rate $1+z_t = \frac{Z_t}{Z_{t-1}}$ follows:

$$\ln z_t = (1-\rho_z)\ln z + \rho_z \ln z_{t-1} + u_{z,t} \tag{18}$$

We assume that the foreign price of commodities $P^*_{X,t}$ follows

$$P^*_{X,t} = \kappa_t P^*_t \tag{19}$$

where P^*_t is the foreign price level and κ_t follows the process:

$$\kappa_t = \exp\left(\tilde{\kappa}_t\right)\left(\frac{1+z^*}{1+z^*_X}\right)^t \tag{20}$$

$$\tilde{\kappa}_t = (1-\rho_\kappa)\kappa + \rho_\kappa \tilde{\kappa}_{t-1} + \varepsilon_{\kappa,t} \tag{21}$$

The law of one price holds and so the domestic price of commodities is:

$$P_{X,t} = S_t P^*_{X,t} \tag{22}$$

and the firm takes this as given when making its production decisions.

1.2.2 Non-tradeable Firms

Non-tradeable firms sell differentiated products, which they produce using the Cobb-Douglas production function:

$$Y_{N,t}(i) = A_t Z_{N,t} K_{N,t}^{\alpha_N}(i) \left(Z_t L_{N,t}(i)\right)^{1-\alpha_N(i)} \tag{23}$$

$Z_{N,t}$ is a stationary sector-specific TFP shock:

$$\begin{aligned} Z_{N,t} &= (1+z_N)^t \tilde{Z}_{N,t} \\ \ln\left(\tilde{Z}_{N,t}\right) &= \rho_N \ln\left(\tilde{Z}_{N,t-1}\right) + u_{N,t} \end{aligned} \tag{24}$$

and Z_t is a labour-augmenting technology shock defined above.
Firms can only change prices at some cost, following a Rotemberg (1982) pricing mechanism:

$$\frac{\psi_N}{2}\left(\frac{P_{N,t}(i)}{\bar{\Pi}^N P_{N,t-1}(i)}-1\right)^2 P_{N,t}Y_{N,t}$$

The output of the non-traded sector is an aggregate of the output of each of the non-traded firms

$$Y_{N,t} \equiv \left(\int_0^1 Y_{N,t}(i)^{\frac{\theta_N-1}{\theta_N}} di\right)^{\frac{\theta_N}{\theta_N-1}}$$

1.2.3 Tradeable Firms

Tradeable firms produce using the Cobb-Douglas production function:

$$Y_{H,t}(i) = A_t Z_{H,t} K_{H,t}^{\alpha_H}(i)\left(Z_t L_{H,t}(i)\right)^{1-\alpha_H} \tag{25}$$

$Z_{H,t}$ is a stationary sector-specific TFP shock:

$$\begin{aligned} Z_{H,t} &= (1+z_H)^t \tilde{Z}_{H,t} \\ \ln\left(\tilde{Z}_{H,t}\right) &= \rho_H \ln\left(\tilde{Z}_{H,t-1}\right) + u_{H,t} \end{aligned} \tag{26}$$

and Z_t is a labour-augmenting technology shock defined above.
Firms can only change prices at some cost, following a Rotemberg (1982) pricing mechanism:

$$\frac{\psi_H}{2}\left(\frac{P_{H,t}(i)}{\bar{\Pi}^H P_{H,t-1}(i)}-1\right)^2 P_{H,t}Y_{H,t}$$

The output of the non-traded sector is an aggregate of the output of each of the non-traded firms

$$Y_{H,t} \equiv \left(\int_0^1 Y_{H,t}(i)^{\frac{\theta_H-1}{\theta_H}} di\right)^{\frac{\theta_H}{\theta_H-1}}$$

1.3 Importing Firms

Importing firms purchase foreign good varieties at the price $\varsigma S_t P_t^*(i)$ and sell them in the domestic market at price $P_{F,t}(i)$. The parameter ς represents a subsidy to imported firms, funded by lump-sum taxation. We set the subsidy equal to $\varsigma = (\theta_f - 1)/\theta_f$, thereby ensuring that markups in this sector are zero in equilibrium.
Importing firms can only change prices at some cost, following a Rotemberg [1982] pricing mechanism:

$$\frac{\psi_F}{2}\left(\frac{P_{F,t}(i)}{\bar{\Pi}^F P_{F,t-1}(i)}-1\right)^2 P_{F,t}Y_{F,t} \tag{27}$$

1.4 Foreign Sector

The rate of foreign goods price inflation is Π_t^*, which follows the process,

$$\ln \Pi_t^* = (1-\rho_{\Pi^*})\ln \Pi^* + \rho_{\Pi^*} \ln \Pi_{t-1}^* + u_{\Pi^*,t} \tag{28}$$

We also assume that foreign demand for the domestically produced tradable, $C^*_{H,t}$, follows the process below;

$$C^*_{H,t} = \gamma^*_{H,t} \left(\frac{P_{H,t}}{S_t P^*_t} \right)^{-\eta^*} Y^*_t \tag{29}$$

where

$$\begin{aligned} Y^*_t &= Z_t (1+z^*)^t \tilde{Y}^*_t \\ \ln\left(\tilde{Y}^*_t\right) &= \rho^*_Y \ln\left(\tilde{Y}^*_t\right) + u^*_t \end{aligned} \tag{30}$$

and $\gamma^*_{H,t}$ follows the process:

$$\gamma^*_{H,t} = \gamma^*_H \left(\frac{1+z_H}{1+z^*} \right)^{t(1-\eta^*)}$$

1.5 Relative Prices and Current Account

In what follows it will be convenient to define a number of relative prices:

$$T_{N,t} = \frac{P_{N,t}}{P_t} \tag{31}$$

$$T_{T,t} = \frac{P_{T,t}}{P_t} \tag{32}$$

$$T^I_{T,t} = \frac{P^I_{T,t}}{P_t} \tag{33}$$

$$T_{H,t} = \frac{P_{H,t}}{P_t} \tag{34}$$

$$T_{F,t} = \frac{P_{F,t}}{P_t} \tag{35}$$

$$T_{I,t} = \frac{P_{I,t}}{P_t} \tag{36}$$

$$T_{X,t} = \frac{P_{X,t}}{P_t} \tag{37}$$

$$T_{F^*,t} = \frac{S_t P^*_t}{P_t} \tag{38}$$

Nominal net exports are given by:

$$NX_t = P_{H,t} C^*_{H,t} + P_{X,t} Y_{X,t} - S_t P^*_t Y_{F,t} \tag{39}$$

And the current account equation is given by:

$$S_t \left(B^*_{t+1} - B^*_t \right) = R^*_{t-1} S_t B^*_t + NX_t \tag{40}$$

1.6 Monetary Policy

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\rho_R} \left[\left(\frac{P_t/P_{t-1}}{\bar{\Pi}} \right)^{\phi_\pi} \left(\frac{Y_t/Y_{t-1}}{Y_t/\bar{Y}_{t-1}} \right)^{\phi_y} \right]^{(1-\rho_R)} \exp(u_{R,t}) \tag{41}$$

1.7 Market Clearing

Investment goods:

$$I_t = J_{H,t} + J_{N,t} + J_{X,t} \tag{42}$$

Labour market:

$$L_t = \left[L_{H,t}^{1+\omega} + L_{N,t}^{1+\omega} + L_{X,t}^{1+\omega}\right]^{\frac{1}{1+\omega}} \tag{43}$$

Non-tradeable goods:

$$Y_{N,t} = C_{N,t} + I_{N,t} + \frac{\psi_N}{2}\left(\frac{\Pi_{N,t}}{\bar{\Pi}^N} - 1\right)^2 Y_{N,t} \tag{44}$$

Tradeables:

$$Y_{H,t} = C_{H,t} + C_{H,t}^* + I_{H,t} + \frac{\psi_H}{2}\left(\frac{\Pi_{H,t}}{\bar{\Pi}^H} - 1\right)^2 Y_{H,t} \tag{45}$$

Imports:

$$Y_{F,t} = C_{F,t} + I_{F,t} + \frac{\psi_F}{2}\left(\frac{\Pi_{F,t}}{\bar{\Pi}^F} - 1\right)^2 Y_{F,t} \tag{46}$$

Nominal GDP:

$$NGDP_t = P_{H,t}Y_{H,t} + P_{N,t}Y_{N,t} + P_{X,t}Y_{X,t} \tag{47}$$

and Real GDP:

$$Y_t = T_H Y_{H,t} + T_N Y_{N,t} + T_X Y_{X,t} \tag{48}$$

2 Derivation of equilibrium conditions

2.1 Households

The household's problem is to choose a sequence for $C_t,\ L_{H,t},\ L_{N,t},\ L_{X,t},\ I_{h,t}, I_{N,t}, I_{X,t},\ K_{H,t+1}, K_{N,t+1}, K_{X,t+1},\ B_{t+1},\ B_{t+1}^*$ to maximise expected lifetime utility subject to the budget constraint and capital accumulation equation. The Lagrangian for this problem is:

$$\begin{aligned}
\mathcal{L} \quad = \quad & E_0 \sum_{t=0}^{\infty} \chi_{t-1} \Bigg\{ \zeta_t \ln\left(C_t - hC_{t-1}\right) - \frac{\xi \epsilon_t^L}{1+\nu}\left[L_{H,t}^{1+\omega} + L_{N,t}^{1+\omega} + L_{X,t}^{1+\omega}\right]^{\frac{1+\nu}{1+\omega}} \\
& + \tilde{\Lambda}_t \Bigg[\left(1 + R_{t-1}\right) B_t + \left(1 + R_{t-1}^*\right) S_t B_t^* + W_{H,t}L_{H,t} + W_{N,t}L_{N,t} + W_{X,t}L_{X,t} \\
& + R_{H,t}^K K_{H,t} + R_{N,t}^K K_{N,t} + R_{X,t}^K K_{X,t} + \Gamma_{N,t} + \Gamma_{H,t} + \Gamma_{X,t} \\
& - P_t C_t - P_{I,t} I_t - B_{t+1} - S_t B_{t+1}^* \Bigg] \\
& + \Phi_{H,t} \left[\left(1-\delta\right) K_{H,t} + V_t \left[1 - \Upsilon\left(\frac{J_{H,t}}{J_{H,t-1}}\right)\right] J_{H,t} - K_{H,t+1} \right] \\
& + \Phi_{N,t} \left[\left(1-\delta\right) K_{N,t} + V_t \left[1 - \Upsilon\left(\frac{J_{N,t}}{J_{N,t-1}}\right)\right] J_{N,t} - K_{N,t+1} \right] \\
& + \Phi_{X,t} \left[\left(1-\delta\right) K_{X,t} + V_t \left[1 - \Upsilon\left(\frac{J_{X,t}}{J_{X,t-1}}\right)\right] J_{X,t} - K_{X,t+1} \right] \Bigg\}
\end{aligned}$$

The first order conditions for the household's problem are:

$$0 = \frac{\zeta_t}{C_t - hC_{t-1}} - hE_t\left\{\beta\frac{\zeta_{t+1}}{C_{t+1} - hC_t}\right\} - \Lambda_t \tag{49}$$

$$0 = -\Lambda_t + E_t\left\{\beta\frac{\Lambda_{t+1}\left(1+R_t\right)P_t}{P_{t+1}}\right\} \tag{50}$$

$$0 = -\Lambda_t + E_t\left\{\beta\frac{\Lambda_{t+1}\left(1+R_t^*\right)P_tS_{t+1}}{P_{t+1}S_t}\right\} \tag{51}$$

$$0 = -Q_{H,t} + E_t\left\{\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left[\frac{R_{H,t+1}^K}{P_{t+1}} + (1-\delta)\,Q_{H,t+1}\right]\right\} \tag{52}$$

$$0 = -Q_{N,t} + E_t\left\{\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left[\frac{R_{N,t+1}^K}{P_{t+1}} + (1-\delta)\,Q_{N,t+1}\right]\right\} \tag{53}$$

$$0 = -Q_{X,t} + E_t\left\{\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left[\frac{R_{X,t+1}^K}{P_{t+1}} + (1-\delta)\,Q_{X,t+1}\right]\right\} \tag{54}$$

$$\begin{aligned} 0 = {} & -T_t^I + Q_{H,t}V_t\left[1 - \Upsilon\left(\frac{J_{H,t}}{J_{H,t-1}}\right) - \Upsilon'\left(\frac{J_{H,t}}{J_{H,t-1}}\right)\frac{J_{H,t}}{J_{H,t-1}}\right] \\ & + E_t\left\{\beta\frac{\Lambda_{t+1}Q_{H,t+1}V_{t+1}}{\Lambda_t}\Upsilon'\left(\frac{J_{H,t+1}}{J_{H,t}}\right)\frac{J_{H,t+1}^2}{J_{H,t}^2}\right\} \end{aligned} \tag{55}$$

$$\begin{aligned} 0 = {} & -T_t^I + Q_{N,t}V_t\left[1 - \Upsilon\left(\frac{J_{N,t}}{J_{N,t-1}}\right) - \Upsilon'\left(\frac{J_{N,t}}{J_{N,t-1}}\right)\frac{J_{N,t}}{J_{N,t-1}}\right] \\ & + E_t\left\{\beta\frac{\Lambda_{t+1}Q_{N,t+1}V_{t+1}}{\Lambda_t}\Upsilon'\left(\frac{J_{N,t+1}}{J_{N,t}}\right)\frac{J_{N,t+1}^2}{J_{N,t}^2}\right\} \end{aligned} \tag{56}$$

$$\begin{aligned} 0 = {} & -T_t^I + Q_{X,t}V_t\left[1 - \Upsilon\left(\frac{I_{x,t}}{J_{X,t-1}}\right) - \Upsilon'\left(\frac{I_{x,t}}{J_{X,t-1}}\right)\frac{J_{X,t}}{J_{X,t-1}}\right] \\ & + E_t\left\{\beta\frac{\Lambda_{t+1}Q_{X,t+1}V_{t+1}}{\Lambda_t}\Upsilon'\left(\frac{J_{X,t+1}}{J_{X,t}}\right)\frac{J_{X,t+1}^2}{J_{X,t}^2}\right\} \end{aligned} \tag{57}$$

and

$$0 = -\zeta_t\xi\epsilon_t^L\left(L_t\right)^{\nu-\omega}L_{H,t}^{\omega} + \Lambda_t\frac{W_{H,t}}{P_t} \tag{58}$$

$$0 = -\zeta_t\xi\epsilon_t^L\left(L_t\right)^{\nu-\omega}L_{N,t}^{\omega} + \Lambda_t\frac{W_{N,t}}{P_t} \tag{59}$$

$$0 = -\zeta_t\xi\epsilon_t^L\left(L_t\right)^{\nu-\omega}L_{X,t}^{\omega} + \Lambda_t\frac{W_{X,t}}{P_t} \tag{60}$$

where $\Lambda_t \equiv \tilde{\Lambda}_tP_t$ is the discount factor on nominal claims and $Q_{j,t} \equiv \Phi_t/\Lambda_t$ is Tobin's Q for sector j.
Demand for each type of consumption good:

$$C_{N,t} = \gamma_{N,t}\left(T_{N,t}\right)^{-\eta}C_t \tag{61}$$

$$C_{T,t} = \gamma_{T,t}\left(T_{T,t}\right)^{-\eta}C_t \tag{62}$$

$$C_{H,t} = \gamma_H\left(T_{H,t}\right)^{-1}C_{T,t} \tag{63}$$

$$C_{F,t} = \gamma_F\left(T_{F,t}\right)^{-1}C_{T,t} \tag{64}$$

and for investment goods:

$$I_{N,t} = \gamma^I_{N,t} \left(\frac{T_{N,t}}{T_{I,t}}\right)^{-1} I_t \tag{65}$$

$$I_{T,t} = \gamma^I_{T,t} \left(\frac{T_{T,t}}{T_{I,t}}\right)^{-1} I_t \tag{66}$$

$$I_{H,t} = \gamma^I_{H} \left(\frac{T_{H,t}}{T_{I,t}}\right)^{-1} I_{T,t} \tag{67}$$

$$I_{F,t} = \gamma^I_{F} \left(\frac{T_{F,t}}{T_{I,t}}\right)^{-1} I_{T,t} \tag{68}$$

2.2 Firms

2.2.1 Commodity Firms

Given prices, commodity firms choose $L_{X,t}$ and $K_{X,t}$ to maximise profits, given by:

$$\Gamma_{X,t} = P_{X,t}Y_{X,t} - W_{X,t}L_{X,t} - R^{KX}_t K_{X,t}$$

The resulting quantities are:

$$\alpha_X \frac{P_{X,t}Y_{X,t}}{K_{X,t}} = R^K_{X,t} \tag{69}$$

$$(1-\alpha_X)\frac{P_{X,t}Y_{X,t}}{L_{X,t}} = W_{X,t} \tag{70}$$

Note that this implies that $\Gamma_{X,t} = 0$.

2.2.2 Tradeable Firms

Profits for tradeable firms are given by:

$$\begin{aligned}\Gamma_{H,t}(i) = P_{H,t}(i)Y_{H,t}(i) - \left(W_{H,t}L_{H,t}(i) + R^K_{H,t}K_{H,t}(i)\right) \\ - \frac{\psi_H}{2}\left(\frac{P_{H,t}(i)}{\bar{\Pi}^H P_{H,t-1}(i)} - 1\right)^2 P_{H,t}Y_{H,t}\end{aligned}$$

The optimality conditions of the cost minimisation problem are:

$$K_{H,t} = \frac{\alpha_H}{1-\alpha_H}\frac{W_{H,t}}{R^K_{H,t}}L_{H,t} \tag{71}$$

$$MC_{H,t} = \left(\frac{1}{1-\alpha_H}\right)^{1-\alpha_H}\left(\frac{1}{\alpha}\right)^{\alpha_H}\frac{W_{H,t}^{1-\alpha_H} {R^K_{H,t}}^{\alpha_H}}{Z_{H,t}Z_t^{1-\alpha_H}P_{H,t}} \tag{72}$$

where MC_t denotes real marginal cost.

When setting prices, the problem of firm i is to choose $P_{H,t}(i)$ to maximise:

$$\sum_{t=0}^{\infty} \frac{\chi_{t-1}\Lambda_t}{P_t}\left(\Gamma_{H,t}(i)\right)$$

subject to the demand constraint that: $Y_{H,t}(i) = \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\theta_H} Y_{H,t}$. Optimal price setting implies that:

$$0 = \frac{\Pi_{H,t}}{\Pi^H}\left(\frac{\Pi_{H,t}}{\Pi^H} - 1\right) + \frac{\theta_H - 1}{\psi} - \frac{\theta_H}{\psi} MC_{H,t} - E_t\left\{\beta \frac{\Lambda_{t+1}}{\Lambda_t}\frac{Y_{H,t+1}}{Y_{H,t}}\frac{T_{H,t+1}}{T_{H,t}}\frac{\Pi_{H,t+1}}{\Pi^H}\left[\frac{\Pi_{H,t+1}}{\Pi^H} - 1\right]\right\} \tag{73}$$

2.2.3 Non-tradeable Firms

Profits for non-tradeable firms are given by:

$$\begin{aligned}\Gamma_{N,t}(i) = P_{N,t}Y_{N,t}(i) - \big(W_{N,t}L_{N,t}(i) + R^K_{N,t}K_{N,t}(i)\big) \\ - \frac{\psi_N}{2}\left(\frac{P_{N,t}(i)}{\bar{\Pi}^N P_{N,t-1}(i)} - 1\right)^2 P_{N,t}Y_{N,t}\end{aligned}$$

The optimality conditions of the cost minimisation problem are:

$$K_{N,t} = \frac{\alpha_N}{1-\alpha_N}\frac{W_{N,t}}{R^K_{N,t}}L_{N,t} \tag{74}$$

$$MC_{N,t} = \left(\frac{1}{1-\alpha_N}\right)^{1-\alpha_N}\left(\frac{1}{\alpha}\right)^{\alpha_N}\frac{W_{N,t}^{1-\alpha_N}{R^K_{N,t}}^{\alpha_N}}{Z_{N,t}Z_t^{1-\alpha_N}P_{N,t}} \tag{75}$$

where $MC_{N,t}$ denotes real marginal cost.
The firm's pricing problem is to choose $P_{N,t}(i)$ to maximise:

$$\sum_{t=0}^{\infty}\frac{\chi_{t-1}\Lambda_t}{P_t}\left[P_{N,t}(i)Y_{N,t}(i) - MC_{N,t}P_tY_{N,t}(i) - \frac{\psi}{2}\left(\frac{P_{N,t}(i)}{\Pi^N P_{N,t-1}(i)} - 1\right)^2 P_{N,t}Y_{N,t}\right]$$

Optimal price setting implies that the inflation rate of non-tradeable goods is given by:

$$0 = \frac{\Pi_{N,t}}{\Pi^N}\left(\frac{\Pi_{N,t}}{\Pi^N} - 1\right) + \frac{\theta_N - 1}{\psi} - \frac{\theta_N}{\psi} MC_{N,t} - E_t\left\{\beta \frac{\Lambda_{t+1}}{\Lambda_t}\frac{T_{N,t+1}}{T_{N,t}}\frac{Y_{N,t+1}}{Y_{N,t}}\left(\frac{\Pi_{N,t+1}}{\Pi^N} - 1\right)\frac{\Pi_{N,t+1}}{\Pi^N}\right\} \tag{76}$$

2.3 Importing Firms

Profits for importing firms are given by

$$\Gamma_{F,t} = (P_{F,t} - \varsigma S_t P_t^*)\, Y_{F,t}(i) - \frac{\psi_F}{2}\left(\frac{P_{F,t}(i)}{\bar{\Pi}^F P_{F,t-1}(i)} - 1\right)^2 P_{F,t}Y_{F,t} \tag{77}$$

Optimal price setting implies that the inflation rate of imported goods is given by:

$$0 = \frac{\Pi_{F,t}}{\Pi^F}\left(\frac{\Pi_{F,t}}{\Pi^F} - 1\right) + \frac{\theta_F - 1}{\psi} - \frac{\theta_F}{\psi}\varsigma S_t P_t^* - E_t\left\{\beta \frac{\Lambda_{t+1}}{\Lambda_t}\frac{T_{F,t+1}}{T_{F,t}}\frac{Y_{F,t+1}}{Y_{F,t}}\left(\frac{\Pi_{F,t+1}}{\Pi^F} - 1\right)\frac{\Pi_{F,t+1}}{\Pi^F}\right\} \tag{78}$$

3 Normalised Equilibrium Conditions

We normalise the variables as follows:

Households

1. $c_t = \frac{C_t}{Z_t}$
2. $c_{N,t} = \frac{C_{N,t}}{Z_t(1+z_N)^t}$
3. $c_{T,t} = \frac{C_t}{Z_t(1+z_T)^t}$
4. $c_{H,t} = \frac{C_{H,t}}{Z_t(1+z_H)^t}$
5. $c_{F,t} = \frac{C_{F,t}}{Z_t(1+z_F)^t}$
6. $\lambda_t = \Lambda_t Z_t$
7. $k_t = \frac{K_t}{Z_t}$
8. $Y_t = \frac{NGDP_t}{P_t Z_t}$
9. $b^*_{t+1} = \frac{S_t B_{t+1}}{P_t GDP_t}$
10. $nx_t = \frac{NX_t}{P_t Z_t}$
11. $\pi_t = \frac{P_t}{P_{t-1}}$
12. $\pi_{N,t} = \frac{P_{N,t}}{P_{N,t-1}}$
13. $\pi_{T,t} = \frac{P_{T,t}}{P_{T,t-1}}$
14. $\pi_{H,t} = \frac{P_{H,t}}{P_{H,t-1}}$
15. $\pi_{F,t} = \frac{P_{F,t}}{P_{F,t-1}}$
16. $\pi_{I,t} = \frac{P_{I,t}}{P_{I,t-1}}$
17. $\pi^I_{T,t} = \frac{P^I_{T,t}}{P^I_{T,t-1}}$
18. $i_t = \frac{I_t}{Z_t(1+z_I)^t}$
19. $v_t = V_t(1+z_I)^t$

Non-tradeable Firms

20. $y_{N,t} = \frac{Y_{N,t}}{Z_t(1+z_N)^t}$
21. $i_{N,t} = \frac{I_{N,t}}{Z_t(1+z_N)^t}$
22. $k_{N,t} = \frac{K_{N,t}}{Z_t}$
23. $r^K_{N,t} = \frac{R^K_{N,t}}{P_t}$
24. $w_{N,t} = \frac{W_{N,t}}{P_t Z_t}$
25. $mc_{N,t} = MC_{N,t}$
26. $j_{N,t} = \frac{J_{N,t}}{(1+z_I)^t Z_t}$
27. $q_{N,t} = Q_{N,t}$

Home-tradeable Firms

28. $y_{H,t} = \frac{Y_{H,t}}{Z_t(1+z_H)^t}$
29. $c^*_{H,t} = \frac{C^*_{H,t}}{Z_t(1+z_H)^t}$
30. $i_{H,t} = \frac{I_{H,t}}{Z_t(1+z_H)^t}$
31. $k_{H,t} = \frac{K_{H,t}}{Z_t}$
32. $r^K_{H,t} = \frac{R^K_{H,t}}{P_t}$
33. $w_{H,t} = \frac{W_{H,t}}{P_t Z_t}$
34. $mc_{H,t} = MC_{H,t}$
35. $\pi_{H,t} = \frac{P_{H,t}}{P_{H,t-1}}$
36. $j_{H,t} = \frac{J_{H,t}}{(1+z_I)^t Z_t}$
37. $q_{H,t} = Q_{H,t}$

Commodity Firms

38. $y_{X,t} = \frac{Y_{X,t}}{Z_t(1+z_X)^t}$
39. $k_{X,t} = \frac{K_{X,t}}{Z_t}$
40. $r^K_{X,t} = \frac{R^K_{X,t}}{P_t}$
41. $w_{X,t} = \frac{W_{X,t}}{P_t Z_t}$
42. $j_{X,t} = \frac{J_{X,t}}{(1+z_I)^t Z_t}$
43. $q_{X,t} = Q_{X,t}$

Relative Prices

44. $\tau_{I,t} = T_{I,t}(1+z_I)^t$
45. $\tau_{N,t} = T_{N,t}(1+z_N)^t$
46. $\tau_{T,t} = T_{T,t}(1+z_T)^t$
47. $\tau_{H,t} = T_{H,t}(1+z_H)^t$
48. $\tau_{F,t} = T_{F,t}(1+z^*)^t$
49. $\tau_{F^*,t} = T_{F^*,t}(1+z^*)^t$
50. $\tau^I_{T,t} = T^I_{T,t}(1+z^*)^t$

Foreign Economy

51. $\pi^*_t = \frac{P^*_t}{P^*_{t-1}}$

Miscellaneous Variables

52. $\Delta s_t = \frac{S_t}{S_{t-1}}$

The Cobb-Douglas assumption for the tradeable consumption bundle implies that the growth rate of the tradeable consumption bundle is $1 + z_T = (1+z_H)^{\gamma_H}(1+z^*)^{\gamma_F}$. A similar relationship holds for the tradeable investment good.
where it is implied by the bundles that:

$$1 + z_T = (1+z_H)^{\gamma_H}(1+z^*)^{\gamma_F}$$

and

$$1 + z^I_T = (1+z_H)^{\gamma^I_H}(1+z^*)^{\gamma^I_F}$$

and

$$1 + z^I = (1+z_v)(1+z^I_T)^{\gamma^I_T}(1+z_N)^{\gamma^I_N}$$

UIP implies that that s staisfies

$$1 + z^* = \frac{\pi}{s\pi^*}$$

and if there is a steady state for $\tau_{I,t}$ then it follows that

$$1 + z_I = \frac{\pi}{\pi_I}$$

Also, note that $z_t = Z_t / Z_{t-1}$. With these normalisations, the equilibrium conditions are as follows. Household optimisation:

$$0 = \frac{\zeta_t z_t}{c_t z_t - h c_{t-1}} - h E_t \left\{ \beta \frac{\zeta_{t+1}}{c_{t+1} z_{t+1} - h c_t} \right\} - \lambda_t \tag{79}$$

$$0 = -\lambda_t + E_t \left\{ \beta \frac{\lambda_{t+1} (1 + r_t)}{z_{t+1} \pi_{t+1}} \right\} \tag{80}$$

$$0 = -\lambda_t + E_t \left\{ \beta \frac{\lambda_{t+1} (1 + r_t^*) s_{t+1}}{z_{t+1} \pi_{t+1}} \right\} \tag{81}$$

$$0 = -q_{H,t} + E_t \left\{ \beta \frac{\lambda_{t+1}}{\lambda_t z_{t+1}} \left[r_{H,t+1}^K + (1 - \delta) q_{H,t+1} \right] \right\} \tag{82}$$

$$0 = -q_{N,t} + E_t \left\{ \beta \frac{\lambda_{t+1}}{\lambda_t z_{t+1}} \left[r_{N,t+1}^K + (1 - \delta) q_{N,t+1} \right] \right\} \tag{83}$$

$$0 = -q_{X,t} + E_t \left\{ \beta \frac{\lambda_{t+1}}{\lambda_t z_{t+1}} \left[r_{X,t+1}^K + (1 - \delta) q_{X,t+1} \right] \right\} \tag{84}$$

$$\begin{aligned} 0 = {} & -\tau_t^I + q_{H,t} v_t \left[1 - \Upsilon \left(\frac{j_{H,t} z_t (1 + z_I)}{j_{H,t-1}} \right) - \Upsilon' \left(\frac{j_{H,t} z_t (1 + z_I)}{j_{H,t-1}} \right) \frac{j_{H,t} z_t (1 + z_I)}{j_{H,t-1}} \right] \\ & + E_t \left\{ \beta \frac{\lambda_{t+1} q_{H,t+1} v_{t+1}}{\lambda_t z_{t+1} (1 + z_I)} \Upsilon' \left(\frac{j_{H,t+1} z_{t+1} (1 + z_I)}{j_{H,t}} \right) \left(\frac{j_{H,t+1} z_{t+1} (1 + z_I)}{j_{H,t}} \right)^2 \right\} \end{aligned} \tag{85}$$

$$\begin{aligned} 0 = {} & -\tau_t^I + q_{N,t} v_t \left[1 - \Upsilon \left(\frac{j_{N,t} z_t (1 + z_I)}{j_{N,t-1}} \right) - \Upsilon' \left(\frac{j_{N,t} z_t (1 + z_I)}{j_{N,t-1}} \right) \frac{j_{N,t} z_t (1 + z_I)}{j_{N,t-1}} \right] \\ & + E_t \left\{ \beta \frac{\lambda_{t+1} q_{N,t+1} v_{t+1}}{\lambda_t z_{t+1} (1 + z_I)} \Upsilon' \left(\frac{j_{N,t+1} z_{t+1} (1 + z_I)}{j_{N,t}} \right) \left(\frac{j_{N,t+1} z_{t+1} (1 + z_I)}{j_{N,t}} \right)^2 \right\} \end{aligned} \tag{86}$$

$$\begin{aligned} 0 = {} & -\tau_t^I + q_{X,t} v_t \left[1 - \Upsilon \left(\frac{j_{X,t} z_t (1 + z_I)}{j_{X,t-1}} \right) - \Upsilon' \left(\frac{j_{X,t} z_t (1 + z_I)}{j_{X,t-1}} \right) \frac{j_{X,t} z_t (1 + z_I)}{j_{X,t-1}} \right] \\ & + E_t \left\{ \beta \frac{\lambda_{t+1} q_{X,t+1} v_{t+1}}{\lambda_t z_{t+1} (1 + z_I)} \Upsilon' \left(\frac{j_{X,t+1} z_{t+1} (1 + z_I)}{j_{X,t}} \right) \left(\frac{j_{X,t+1} z_{t+1} (1 + z_I)}{j_{X,t}} \right)^2 \right\} \end{aligned} \tag{87}$$

and

$$0 = -\zeta_t \xi \epsilon_t^L (L_t)^{\nu - \omega} L_{H,t}^{\omega} + \lambda_t w_{H,t} \tag{88}$$

$$0 = -\zeta_t \xi \epsilon_t^L (L_t)^{\nu - \omega} L_{N,t}^{\omega} + \lambda_t w_{N,t} \tag{89}$$

$$0 = -\zeta_t \xi \epsilon_t^L (L_t)^{\nu - \omega} L_{X,t}^{\omega} + \lambda_t w_{X,t} \tag{90}$$

Capital accumulation:

$$0 = k_{H,t+1} E_t \left\{ z_{t+1} \right\} - (1-\delta)\, k_{H,t} - v_t \left[1 - \Upsilon \left(\frac{j_{H,t}}{j_{H,t-1}} z_t (1+z_I) \right) \right] j_{H,t} \tag{91}$$

$$0 = k_{N,t+1} E_t \left\{ z_{t+1} \right\} - (1-\delta)\, k_{N,t} - v_t \left[1 - \Upsilon \left(\frac{j_{N,t}}{j_{N,t-1}} z_t (1+z_I) \right) \right] j_{N,t} \tag{92}$$

$$0 = k_{X,t+1} E_t \left\{ z_{t+1} \right\} - (1-\delta)\, k_{X,t} - v_t \left[1 - \Upsilon \left(\frac{j_{X,t}}{j_{X,t-1}} z_t (1+z_I) \right) \right] j_{X,t} \tag{93}$$

Price and inflation indices:

$$\pi_t = \left[\gamma_N \left(\pi_{N,t} \tau_{N,t-1} (1+z_N) \right)^{1-\eta} + \gamma_T \left(\pi_{T,t} \tau_{T,t-1} (1+z_T) \right)^{1-\eta} \right]^{\frac{1}{1-\eta}} \tag{94}$$

$$\pi_{T,t} = \pi_{H,t}^{\gamma_H} \pi_{F,t}^{\gamma_F} \tag{95}$$

$$\pi_t^I = {\pi_{T,t}^I}^{\gamma_T^I} \pi_{N,t}^{\gamma_N^I} \tag{96}$$

$$\pi_{T,t}^I = \pi_{H,t}^{\gamma_H^I} \pi_{F,t}^{\gamma_F^I} \tag{97}$$

Consumer demand:

$$c_{N,t} = \gamma_N \left(\tau_{N,t} \right)^{-\eta} c_t \tag{98}$$

$$c_{T,t} = \gamma_T \left(\tau_{T,t} \right)^{-\eta} c_t \tag{99}$$

$$c_{H,t} = \gamma_H \gamma_T \left(\tau_{H,t} \right)^{-1} \left(\tau_{T,t} \right)^{1-\eta} c_t \tag{100}$$

$$c_{F,t} = \gamma_F \gamma_T \left(\tau_{F,t} \right)^{-1} \left(\tau_{T,t} \right)^{1-\eta} c_t \tag{101}$$

Investment demand:

$$i_{N,t} = \gamma_N^I \left(\frac{\tau_{N,t}}{\tau_{I,t}} \right)^{-1} i_t \tag{102}$$

$$i_{T,t} = \gamma_T^I \left(\frac{\tau_{T,t}^I}{\tau_{I,t}} \right)^{-1} i_t \tag{103}$$

$$i_{H,t} = \gamma_H^I \gamma_T^I \left(\frac{\tau_{H,t}}{\tau_{T,t}^I} \right)^{-1} \left(\frac{\tau_{T,t}^I}{\tau_{I,t}} \right)^{-1} i_t \tag{104}$$

$$i_{F,t} = \gamma_F^I \gamma_T^I \left(\frac{\tau_{F,t}}{\tau_{T,t}^I} \right)^{-1} \left(\frac{\tau_{T,t}^I}{\tau_{I,t}} \right)^{-1} i_t \tag{105}$$

Production:

$$y_{X,t} = a_t \tilde{Z}_{X,t} k_{X,t}^{\alpha_X} l_{X,t}^{1-\alpha_X} \tag{106}$$

$$y_{H,t} = a_t \tilde{Z}_{H,t} k_{H,t}^{\alpha_X} l_{H,t}^{1-\alpha_H} \tag{107}$$

$$y_{N,t} = a_t \tilde{Z}_{N,t} k_{N,t}^{\alpha_N} l_{N,t}^{1-\alpha_N} \tag{108}$$

Tradeable firms:

$$0 = k_{H,t} - \frac{\alpha_H}{1-\alpha_H}\frac{w_{H,t}l_{H,t}}{r^K_{H,t}} \tag{109}$$

$$0 = mc_{H,t} - \left(\frac{1}{1-\alpha_H}\right)^{1-\alpha_H}\left(\frac{1}{\alpha_H}\right)^{\alpha_H}\frac{w_{H,t}^{1-\alpha_H}{r^K_{H,t}}^{\alpha_H}}{\tau_{H,t}\tilde{Z}_{H,t}} \tag{110}$$

$$0 = \frac{\pi_{H,t}}{\pi_H}\left(\frac{\pi_{H,t}}{\pi_H}-1\right) + \frac{\theta_H - 1}{\psi_H} - \frac{\theta_H}{\psi_H}mc_{H,t} - E_t\left\{\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{y_{H,t+1}}{y_{H,t}}\frac{\tau_{H,t+1}}{\tau_{H,t}}\frac{\pi_{H,t+1}}{\pi_H}\left[\frac{\pi_{H,t+1}}{\pi_H}-1\right]\right\} \tag{111}$$

Non-tradeable firms:

$$0 = k_{N,t} - \frac{\alpha_N}{1-\alpha_N}\frac{w_{N,t}l_{N,t}}{r^K_{N,t}} \tag{112}$$

$$0 = mc_{N,t} - \left(\frac{1}{1-\alpha_N}\right)^{1-\alpha_N}\left(\frac{1}{\alpha_N}\right)^{\alpha_N}\frac{w_{N,t}^{1-\alpha_N}{r^K_{N,t}}^{\alpha_N}}{\tau_{N,t}\tilde{Z}_{N,t}} \tag{113}$$

$$0 = \frac{\pi_{N,t}}{\pi_N}\left(\frac{\pi_{N,t}}{\pi_N}-1\right) + \frac{\theta_N - 1}{\psi_N} - \frac{\theta_N}{\psi_N}mc_{N,t} - E_t\left\{\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{y_{N,t+1}}{y_{N,t}}\frac{\tau_{N,t+1}}{\tau_{N,t}}\frac{\pi_{N,t+1}}{\pi_N}\left[\frac{\pi_{N,t+1}}{\pi_N}-1\right]\right\} \tag{114}$$

Commodity firms:

$$0 = \alpha_X\frac{\tau_{X,t}y_{X,t}}{k_{X,t}} - r^K_{X,t} \tag{115}$$

$$0 = (1-\alpha_X)\frac{\tau_{X,t}y_{X,t}}{l_{X,t}} - w_{X,t} \tag{116}$$

Importing firms

$$0 = \frac{\pi_{F,t}}{\pi_F}\left(\frac{\pi_{F,t}}{\pi_F}-1\right) + \frac{\theta_F - 1}{\psi_F} - \frac{\theta_F}{\psi_F}mc_{F,t} - E_t\left\{\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{y_{F,t+1}}{y_{F,t}}\frac{\tau_{F,t+1}}{\tau_{F,t}}\frac{\pi_{F,t+1}}{\pi_F}\left[\frac{\pi_{F,t+1}}{\pi_F}-1\right]\right\} \tag{117}$$

$$mc_{F,t} = \varsigma\left(\frac{\tau_{F^*,t}}{\tau_{F,t}}\right) \tag{118}$$

Law of one price:

$$0 = \tau_{X,t} - \kappa_t\tau_{F^*,t} \tag{119}$$

Relative Prices:

$$0 = \frac{\tau_{N,t}}{\tau_{N,t-1}} - \frac{\pi_{N,t}(1+z_N)}{\pi_t} \tag{120}$$

$$0 = \frac{\tau_{T,t}}{\tau_{T,t-1}} - \frac{\pi_{T,t}(1+z_T)}{\pi_t} \tag{121}$$

$$0 = \frac{\tau^I_{T,t}}{\tau^I_{T,t-1}} - \frac{\pi^I_{T,t}(1+z^I_T)}{\pi_t} \tag{122}$$

$$0 = \frac{\tau_{H,t}}{\tau_{H,t-1}} - \frac{\pi_{H,t}(1+z_H)}{\pi_t} \tag{123}$$

$$0 = \frac{\tau_{F,t}}{\tau_{F,t-1}} - \frac{\pi_{F,t}(1+z^*)}{\pi_t} \tag{124}$$

$$0 = \frac{\tau_{F^*,t}}{\tau_{F^*,t-1}} - \frac{\Delta s_t \pi^*_t(1+z^*)}{\pi_t} \tag{125}$$

$$0 = \frac{\tau_{I,t}}{\tau_{I,t-1}} - \frac{\pi_{I,t}(1+z_I)}{\pi_t} \tag{126}$$

Foreign sector:

$$c^*_{H,t} = \gamma^*_H \left(\frac{\tau_{H,t}}{\tau_{F^*,t}}\right)^{-\eta^*} \tilde{Y}^*_t \tag{127}$$

Market clearing:

$$0 = y_{N,t} - c_{N,t} - i_{N,t} - \frac{\psi_N}{2}\left(\frac{\pi_{N,t}}{\bar{\pi}^N} - 1\right)^2 y_{N,t} \tag{128}$$

$$0 = y_{H,t} - c_{H,t} - c^*_{H,t} - i_{H,t} - \frac{\psi_H}{2}\left(\frac{\pi_{H,t}}{\bar{\pi}^H} - 1\right)^2 y_{H,t} \tag{129}$$

$$0 = y_{F,t} - c_{F,t} - i_{F,t} - \frac{\psi_F}{2}\left(\frac{\pi_{F,t}}{\pi^F} - 1\right)^2 y_{F,t} \tag{130}$$

$$0 = i_t - j_{n,t} - j_{h,t} - j_{x,t} \tag{131}$$

$$0 = l_t - \left[l^{1+\omega}_{H,t} + l^{1+\omega}_{N,t} + l^{1+\omega}_{X,t}\right]^{\frac{1}{1+\omega}} \tag{132}$$

$$0 = gdp_t - \tau_{H,t}y_{H,t} - \tau_{N,t}y_{N,t} - \tau_{X,t}y_{X,t} \tag{133}$$

$$0 = nx_t - \tau_{H,t}\frac{c^*_{H,t}}{gdp_t} + \tau_{F,t}\left(\frac{c_{F,t} - i_{F,t}}{gdp_t}\right) + \tau_{X,t}\frac{y_{X,t}}{gdp_t} \tag{134}$$

$$0 = b^*_{t+1} - \frac{b^*_t r^*_t gdp_{t-1}}{\pi_t gdp_t z_t} - nx_t \tag{135}$$

Interest rates and monetary policy:

$$1 + r^F_t = (1 + r^*_t)\exp(-\psi_b\,(b^*_t - b^*) + \tilde{\psi}_t) \tag{136}$$

$$\frac{1+r_t}{1+r} = \left(\frac{1+r_{t-1}}{1+r}\right)^{\rho_R}\left[\left(\frac{\pi_t}{\Pi}\right)^{\phi_\pi}\left(\frac{y_t z_t}{y_{t-1}\bar{z}}\right)^{\phi_y}\right]^{(1-\rho_R)} \exp(u_{R,t}) \tag{137}$$

4 Steady State

Household optimisation:

$$0 = \frac{z - h\beta}{(z-h)} - c\lambda \tag{138}$$
$$0 = 1 + r - \frac{\pi z}{\beta} \tag{139}$$
$$0 = s - \frac{1+r}{1+r^*} \tag{140}$$
$$0 = -q_H + \frac{\beta}{z}\left[r_H^K + (1-\delta)q_H\right] \tag{141}$$
$$0 = -q_N + \frac{\beta}{z}\left[r_N^K + (1-\delta)q_N\right] \tag{142}$$
$$0 = -q_X + \frac{\beta}{z}\left[r_X^K + (1-\delta)q_X\right] \tag{143}$$
$$0 = -\tau^I + q_H v \tag{144}$$
$$0 = -\tau^I + q_N v \tag{145}$$
$$0 = -\tau^I + q_X v \tag{146}$$

and

$$0 = -\xi l^{\nu-\omega} l_H^{\omega} + \lambda w_H \tag{147}$$
$$0 = -\xi l^{\nu-\omega} l_N^{\omega} + \lambda w_N \tag{148}$$
$$0 = -\xi l^{\nu-\omega} l_X^{\omega} + \lambda w_X \tag{149}$$

Capital accumulation:

$$0 = k_H(z - 1 + \delta) - v j_H \tag{150}$$
$$0 = k_N(z - 1 + \delta) - v j_N \tag{151}$$
$$0 = k_X(z - 1 + \delta) - v j_X \tag{152}$$
$$\tag{153}$$

Inflation indicies:

$$\pi = \left[\gamma_N\left(\pi_N\tau_N(1+z_N)\right)^{1-\eta} + \gamma_T\left(\pi_T\tau_T(1+z_T)\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \tag{154}$$
$$\pi_T\tau_T = \left(\pi_H\tau_H\right)^{\gamma_H}\left(\pi_F\tau_F\right)^{\gamma_F} \tag{155}$$
$$\pi^I\tau_I = \left(\pi_T^I\tau_T^I\right)^{\gamma_H^I}\left(\pi_N\tau_N\right)^{\gamma_N^I} \tag{156}$$
$$\pi_T^I\tau_T^I = \left(\pi_H\tau_H\right)^{\gamma_H^I}\left(\pi_F\tau_F\right)^{\gamma_F^I} \tag{157}$$

Consumer demand:

$$c_N = \gamma_N \left(\tau_N\right)^{-\eta} c \tag{158}$$
$$c_T = \gamma_T \left(\tau_T\right)^{-\eta} c \tag{159}$$
$$c_H = \gamma_H \gamma_T \left(\tau_H\right)^{-1} \left(\tau_T\right)^{1-\eta} c \tag{160}$$
$$c_F = \gamma_F \gamma_T \left(\tau_F\right)^{-1} \left(\tau_T\right)^{1-\eta} c \tag{161}$$

Investment demand:

$$i_N = \gamma_N^I \left(\frac{\tau_N}{\tau_I}\right)^{-1} i \tag{162}$$
$$i_T = \gamma_T^I \left(\frac{\tau_T^I}{\tau_I}\right)^{-1} i \tag{163}$$
$$i_H = \gamma_H^I \left(\frac{\tau_H}{\tau_T^I}\right)^{-1} i_T \tag{164}$$
$$i_F = \gamma_F^I \left(\frac{\tau_F}{\tau_T^I}\right)^{-1} i_T \tag{165}$$

Production:

$$y_X = k_X^{\alpha} l_X^{1-\alpha} \tag{166}$$
$$y_H = k_H^{\alpha} l_H^{1-\alpha} \tag{167}$$
$$y_N = k_N^{\alpha} l_N^{1-\alpha} \tag{168}$$

Tradeable firms:

$$0 = k_H - \frac{\alpha_H}{1-\alpha_H} \frac{w_H l_H}{r_H^K} \tag{169}$$
$$0 = mc_H - \left(\frac{1}{1-\alpha_H}\right)^{1-\alpha_H} \left(\frac{1}{\alpha_H}\right)^{\alpha_H} \frac{w_H^{1-\alpha_H} {r_H^K}^{\alpha_H}}{\tau_H} \tag{170}$$
$$0 = \frac{\theta_H - 1}{\theta_H} - mc_{H,t} \tag{171}$$

Non-tradeable firms:

$$0 = k_N - \frac{\alpha_N}{1-\alpha_N} \frac{w_N l_N}{r_N^K} \tag{172}$$
$$0 = mc_N - \left(\frac{1}{1-\alpha_N}\right)^{1-\alpha_N} \left(\frac{1}{\alpha_N}\right)^{\alpha_N} \frac{w_N^{1-\alpha_N} {r_N^K}^{\alpha_N}}{\tau_N} \tag{173}$$
$$0 = \frac{\theta_N - 1}{\theta_N} - mc_N \tag{174}$$

Commodity firms:

$$0 = \alpha_X \frac{\tau_X y_X}{k_X} - r_X^K \tag{175}$$
$$0 = \left(1-\alpha_X\right) \frac{\tau_X y_X}{l_X} - w_X \tag{176}$$

Importing firms

$$0 = \frac{\theta_F - 1}{\theta_F} - mc_F \tag{177}$$

$$0 = mc_F - \varsigma\left(\frac{\tau_{F^*}}{\tau_F}\right) \tag{178}$$

Law of one price:

$$0 = \tau_X - \kappa\tau_{F^*} \tag{179}$$

Relative Prices:

$$0 = 1 - \frac{\pi_N(1 + z_N)}{\pi} \tag{180}$$

$$0 = 1 - \frac{\pi_T(1 + z_T)}{\pi} \tag{181}$$

$$0 = 1 - \frac{\pi_T^I(1 + z_T^I)}{\pi} \tag{182}$$

$$0 = 1 - \frac{\pi_H(1 + z_H)}{\pi} \tag{183}$$

$$0 = 1 - \frac{\Delta s\pi^*(1 + z^*)}{\pi} \tag{184}$$

$$0 = 1 - \frac{\pi_F(1 + z^*)}{\pi} \tag{185}$$

$$0 = 1 - \frac{\pi_I(1 + z_I)}{\pi} \tag{186}$$

Market clearing:

$$0 = y_N - c_N - i_N \tag{187}$$

$$0 = y_H - c_H - c_H^* - i_H \tag{188}$$

$$0 = y_F - c_F - i_F \tag{189}$$

$$0 = i - j_h - j_n - j_x \tag{190}$$

$$0 = l - \left[l_H^{1+\omega} + l_N^{1+\omega} + l_X^{1+\omega}\right]^{\frac{1}{1+\omega}} \tag{191}$$

$$0 = gdp - \tau_H y_H - \tau_N y_N - \tau_X y_X \tag{192}$$

$$0 = nx - \tau_H \frac{c_H^*}{gdp} - \tau_F\left(\frac{yF}{gdp}\right) + \tau_X \frac{y_X}{gdp} \tag{193}$$

$$0 = b^* - \frac{b^* r^*}{\pi z} - nx \tag{194}$$

Miscellaneous equations:

$$b = b^* \tag{195}$$

$$\pi = \Pi \tag{196}$$

5 The Posterior Sampler

To simulate from the joint posterior of the structural parameters and the date breaks, $p(\vartheta, \mathbf{T}|Y)$, we use the Metropolis-Hastings algorithm following a strategy similar to Kulish et al. [2014]. As we have continuous and discrete parameters we modify the standard setup for Bayesian estimation of DSGE models. We separate the parameters into two blocks: date breaks and structural parameters. To be clear, though, the sampler delivers draws from the joint posterior of both sets of parameters.

The first block of the sampler is for the date breaks, $\mathbf{T}$. As is common in the literature on structural breaks (Bai and Perron [1998]), we set the trimming parameter to 25 per cent of the sample size so that the minimum length of a segment has 20 observation. Within the feasible range we draw from a uniform proposal density and randomize *which* particular date break in $\mathbf{T}$ to update. This approach is motivated by the randomized blocking scheme developed for DSGE models in Chib and Ramamurthy [2010].

The algorithm for drawing for the date breaks block is as follows: Initial values of the date breaks, $\mathbf{T}_0$, and the structural parameters, ϑ_0, are set. Then, for the j^{th} iteration, we proceed as follows:

1. randomly sample *which* date break to update from a discrete uniform distribution with support ranging from one to the total number of breaks, in our case two.

2. randomly sample the corresponding elements of the proposed date breaks, $\mathbf{T}^{'}_{j}$, from a discrete uniform distribution $[T_{min}, T_{max}]$ and set the remaining elements to their values in $\mathbf{T}_{j-1}$

3. calculate the acceptance ratio $\alpha_j^{\mathbf{T}} \equiv \frac{p(\vartheta_{j-1}, \mathbf{T}^{'}_{j}|\mathcal{Y})}{p(\vartheta_{j-1}, \mathbf{T}_{j-1}|\mathcal{Y})}$

4. accept the proposal with probability $\min\{\alpha_j^{\mathbf{T}}, 1\}$, setting $\mathbf{T}_j = \mathbf{T}^{'}_{j}$, or $\mathbf{T}_{j-1}$ otherwise.

The second block of the sampler is for the n_ϑ structural parameters.[1] It follows a similar strategy to the date-breaks-block described above - we randomize over the number and which parameters to possibly update at each iteration. The proposal density is a multivariate Student's $t-$ distribution.[2] Once again, for the j^{th} iteration we proceed as follows:

1. randomly sample the *number* of parameters to update from a discrete uniform distribution $[1, n_\vartheta]$

2. randomly sample without replacement *which* parameters to update from a discrete uniform distribution $[1, n_\vartheta]$

3. construct the proposed $\vartheta^{'}_{j}$ by drawing the parameters to update from a multivariate Student's $t-$ distribution with 10 degrees of freedom and with location set at the corresponding elements of ϑ_{j-1}, scale matrix based on the corresponding elements of the negative inverse Hessian at the posterior mode multiplied by a tuning parameter $\iota = 0.15$.

4. calculate the acceptance ratio $\alpha_j^{\vartheta} \equiv \frac{p(\vartheta^{'}_{j}, \mathbf{T}_j|\mathcal{Y})}{p(\theta_{j-1}, \mathbf{T}_j|\mathcal{Y})}$ or set $\alpha_j^{\vartheta} = 0$ if the proposed $\vartheta^{'}_{j}$ includes inadmissible values (e.g. a proposed negative value for the standard deviation of a shock or autoregressive parameters above unity) preventing calculation of $p(\vartheta^{'}_{j}, \mathbf{T}_j|Y)$

5. accept the proposal with probability $\min\{\alpha_j^{\vartheta}, 1\}$, setting $\vartheta_j = \vartheta^{'}_{j}$, or ϑ_{j-1} otherwise.

We use this multi-block algorithm to construct a chain of 575,000 draws from the joint posterior, $p(\vartheta, \mathbf{T}|Y)$, throwing out the first 25 per cent as burn-in. Trace plots show that the sampler mixes well.

[1] In our application $n_\vartheta = 37$.

[2] The hessian of the proposal density is computed at the mode of the structural parameters.

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