A Model of Secular Stagnation: Theory and Quantitative Evaluation*

Gauti B. Eggertsson[†] Neil R. Mehrotra[‡] Jacob Robbins [§]

This version: December 7, 2016

Abstract

We formalize the idea of secular stagnation and evaluate it quantitatively. Secular stagnation can arise if there is a persistent decline in the natural rate of interest, which, if severe enough, results in a chronically binding zero lower bound. Our theory carries fundamentally different implications for monetary/fiscal policy and output/inflation dynamics relative to the standard New Keynesian model. Slow-moving forces at work across many advanced economies: low productivity growth, low population growth, higher life expectancy, falling prices of capital goods, increasing inequality, and deleveraging can generate a secular stagnation. Using a quantitative, 56 generation lifecycle model calibrated to US data, we provide numerical experiments in which these forces are strong enough to generate a natural rate of interest for the US from -1.5% to -2% in the stationary equilibrium. In this scenario, given the current inflation target and fiscal policy configuration, our model predicts the zero lower bound is likely to remain problematic for the foreseeable future.

Keywords: Secular stagnation, monetary policy, zero lower bound

JEL Classification: E31, E32, E52

*This paper replaces and earlier version of the paper released in 2014 under the the title "A Model of Secular Stagnation". We thank Olivier Blanchard, John Cochrane, Michael Dotsey, Jesus Fernandez-Villaverde, Pablo Guerron-Quintana, Benjamin Keen, Pat Kehoe, John Leahy, Emi Nakamura, and Paolo Pesenti for helpful discussions and conference and seminar participants at the Bank of Canada, Bank of England, Bank for International Settlements, Bank of Japan, Bank of Finland, Boston University, Brown, Central Bank of Austria, Cornell, Chicago Booth, European Central Bank, the Federal Reserve Banks of Boston, Dallas, Minneapolis and New York, Harvard, LSE, LUISS Guido Carli, NBER Summer Institute MEFM and EFG meetings, Northwestern, Princeton, Royal Economic Society meetings, and UTDT for comments. We would also like to thank Alex Mechanick for excellent research assistance.

[†]Brown University, Department of Economics, e-mail: gauti_eggertsson@brown.edu

[‡]Brown University, Department of Economics, e-mail: neil_mehrotra@brown.edu

[§]Brown University, Department of Economics, e-mail: jacob_Robbins@brown.edu

1 Introduction

Nearly a decade into the Great Depression, the President of the American Economic Association, Alvin Hansen, delivered a disturbing message in his Presidential Address to the Association (see Hansen (1939)). He suggested that the Great Depression might just be the start of a new era of ongoing unemployment and economic stagnation without any expectation of returning to full employment; an era of "secular stagnation". Hansen was concerned that an absence of new investment opportunities and a decline in the population birth rate had led to an oversupply of savings relative to investment demand. Ultimately, Hansen's predictions were a bust. US entry into the Second World War lead to a massive expansion in government expenditure, eliminating a demand shortfall that had persisted for over a decade. Moreover, the baby boom following the Second World War drastically shifted the population dynamics in the US, thus erasing the demographic problems that Hansen had foreseen.

The current economic environment gives new breath to Hansen's secular stagnation hypothesis. In influential remarks at the IMF in 2013, Lawrence Summers resurrected the secular stagnation hypothesis arguing that advanced economies in the wake of the Great Recession are plagued by many of the same ailments identified by Hansen: elevated unemployment, output below trend, low interest rates and inflation below target. Indeed, secular stagnation conditions may have existed prior to 2008, but the tech bubble in the late 1990s and the subsequent housing bubble in the early 2000s masked their effects. In Summers' words, we may have found ourselves in a situation in which the natural rate of interest - the short-term real interest rate consistent with full employment - is permanently negative (see Summers (2013)). And a permanently negative natural rate of interest has profound implications for the conduct of monetary, fiscal, and financial stability policy today.

Despite the prominence of Summers' discussion of the secular stagnation hypothesis and a flurry of commentary that followed it (see e.g. Krugman (2013), Taylor (2014), Delong (2014) for a few examples), there has not, to the best of our knowledge, been any attempt to formally model this idea - to write down an explicit model in which the natural rate of interest may be persistently negative. The goal of this paper is to fill this gap. We show how permanently negative natural interest rates can emerge in a three-period OLG model in the spirit of Samuelson (1958). Adding nominal frictions, we then show how negative natural interest rates translate into secular stagnation. We define secular stagnation as the combination of interest rate close to zero, inflation below target and the presence of an output gap on a persistent basis. Finally, we extend this framework to a 56 generation quantitative lifecycle model with capital and highlight a variety of forces that can account for low interest rates in the US. The bottom line of our quantitive exercise is that under plausible parameterization the natural rate of interest rate will remain negative in the foreseeable future. This is a key necessary condition for a secular stagnation environment. Our calibration suggests danger of frequent and recurring ZLB episodes going forward in the US as has been observed in Japan in recent decades.

It may seem somewhat surprising that the idea of secular stagnation has not already been studied in detail in the recent literature on the liquidity trap. This literature already invites the possibility that the zero bound on the nominal interest rate is binding for some period of time due to a drop in the natural rate of interest. The reason for this omission, we suspect, is that secular stagnation does not emerge naturally from the current vintage of models in use in the literature. This, however - and perhaps unfortunately - has less to do with economic reality than with the limitations of these models. Most analyses of the current crisis takes place within representative agent models (see e.g. Krugman (1998), Eggertsson and Woodford (2003), Christiano, Eichenbaum and Rebelo (2011) and Werning (2012) for a few well known examples) where the long run real interest rate is directly determined by the discount factor of the representative agent. Zero lower bound episodes are caused by temporary shocks to the discount factor which must revert back to a positive long-run level.[1] Our model is fundamentally different from this earlier class of models. With an overlapping generations structure, the discount factor is no longer the sole determinant of the natural rate of interest and a negative natural rate that lasts for an arbitrarily long time is now a possibility.

Existing narratives of the Great Recession in US have emphasized the effect of debt deleveraging caused by the housing crisis on restraining aggregate demand and causing the ZLB to bind (see, for example, Mian and Sufi (2012), Eggertsson and Krugman (2012), Guerrieri and Lorenzoni (2011), and Mehrotra (2012)). In these models, even a permanent shock that tightens credit to financially constrained households results in only a temporary ZLB episode - once the deleveraging cycle runs its course, the natural rate of interest rate rises, aggregate demand rises, and the ZLB episode ends. In stark contrast, in our model, a deleveraging cycle need not culminate in a rise in the natural rate. Rather, the borrowers in our model, by taking on less debt in their earlier years, have greater disposable income in the middle saving period of their life. This actually worsens the saving glut putting further downward pressure on the natural rate of interest. In our model, there is no guarantee of a recovery once the deleveraging cycle ends.

In addition to allowing for the possibility of arbitrarily long periods of a negative natural rate, the OLG structure of our quantitative model allows us to consider a host of new forces that affect the natural rate of interest. A slowdown in population growth or increase in life expectancy puts downward pressure on the natural rate. Rising income inequality or a fall in the relative price of investment goods may also reduce the natural rate. Finally, slowdown in productivity plays an important role. Thus, our model shows how arguably temporary shocks like debt deleveraging

[1] A permanent shock to the households discount factor is not possible since the maximization problem of the representative household is no longer well defined. One alternative in the representative agent framework is large enough uncertainty to make the risk-free rate negative, see Abel et al. (1989).

can interact with long run forces that appear to be at work across many advanced economies to lower the natural rate of interest. This interplay between long-run forces and business cycle shocks is a key advantage of our framework relative to the existing class of ZLB models.

Recent literature has emphasized that existing New Keynesian models of the zero lower bound tend to generate inflation and output dynamics at odds with recent ZLB episodes, predicting sharp collapses in output and deflationary spirals for particularly long lasting ZLB episodes. By contrast, our model does not suffer from these dynamics. In a secular stagnation steady state, inflation is persistently below target and output falls below trend, which closely matches the observed dynamics of output, inflation and interest rates in the US, Eurozone and Japan. Related to deflationary spirals, our model also does not suffer from the forward guidance puzzle, the idea that monetary policy very far in the future has ”implausibly large effect” (see Del Negro, Giannoni and Patterson (2012)). The intertemporal IS equation in our model is not as ”forward-looking” and displays endogenous discounting as in McKay, Nakamura and Steinsson (2015) due to lifecycle considerations rather than occasionally binding borrowing constraints.

The policy implications of our framework differ in marked ways from the standard paradigm. Perhaps most importantly a policy of waiting for a ZLB episode to end is not an option in a secular stagnation driven by highly persistent forces like a slowdown in population growth - there is no deus ex machina for recovery. This has important implications for monetary policy. Unlike Eggertsson and Woodford (2003), forward guidance is an ineffective policy in our framework for two reasons. First, agents do not necessarily anticipate a future date at which the ZLB is not binding and, second, because the intertemporal IS equation is not as ”forward-looking.” Raising the inflation target may be an option to accommodate a negative natural rate of interest, but this policy suffers from two drawbacks. First, an increase in the inflation target must be sufficiently large. For example, if the natural rate of interest is negative 4%, the inflation target must be 4% or higher. Small changes in the inflation target have no effect, formalizing Krugman idea of the ”law of the excluded middle” or ”timidity trap.” Second, even with a sufficiently large increase in the inflation target, the secular stagnation steady is also an equilibrium.

Fiscal policy is considerably more effective in addressing problems raised secular stagnation and does not suffer from the issue of multiplicity. An increase in government spending or, since our model is not Ricardian, an increase in public debt, can raise the natural rate of interest and circumvent the zero lower bound. However, fiscal policy operates in a more subtle manner than in the standard New Keynesian accounts. Increases in government spending can carry zero or negative multipliers in our model depending on which generations bear the tax. The key for fiscal policy to be successful is that it must reduce the oversupply of saving. Fiscal policy that instead leaves saving unchanged or increases desired saving by, for example, reducing future income, could exacerbate a secular stagnation.

While the first main contribution of this paper is an analytic framework that lays out the theoretical ingredients needed to formally characterize secular stagnation, the second main contribution is building a medium scale quantitative model to quantitatively explore whether persistently negative natural interest rates are quantitatively plausible, the key necessary condition for an environment of secular stagnation in which the ZLB is chronically binding. We build a 56 period medium scale OLG model, and calibrate our model to match the US economy in 2015, with a target natural rate of interest of -1.1% to match current interest rate data. Our quantitative model is able to generate a permanently *negative* natural rate of interest using parameters that are standard from the macro literature, and matches key moments from US data. The lynchpin of these negative natural rates are an aging population, low fertility, and sluggish productivity growth. While we hope we are wrong, like Hansen was, and this trend will reverse itself, if current projections for fertility and productivity hold our analysis suggests that the natural rate of interest will be low or negative for the foreseeable future. Of course, a negative natural rate of interest does in steady state does not exclude the possibility that we see a temporary rise in nominal interest rate due to temporary factors. Instead it suggests that there are plausible conditions under which one should expect recurrent and chronic ZLB episodes going forward that can be of arbitrary duration.

There has been lively debate on if the US economy already closed its output gap in 2015 (see e.g. Stock and Watson that claim it is close to zero, and Hall (2014) that estimates it to be substantial). We do not contribute to this debate. Instead, we conduct two numerical experiments taking both sides of the debate: in the first one, our benchmark calibration, we assume that the US economy in 2015 was operating without any output gap. Our second exercise models the US economy with a substantial output gap. This is all the more relevant when one considers Europe and Japan, where there is less disagreement about the presence of output gap. Our quantitative model can generate a realistic ZLB episode consistent with either estimate of the current output gap in the US.

Finally, we use our quantitative model to explore the decline in real interest rates seen over the past forty years and to make projections about the future path of real interest rates. In the United States and most other developed nations, the real interest rate has decline substantially over the past forty years. The real Federal Funds rate rises from 2.6% into the early 1980s and shows a declining trend over the subsequent 35 years to -1.5%. Our quantitative analysis incorporates both Hansen's and Summers's determinations of secular stagnation, and examines changes in fertility, the rate of productivity growth, and a decline in the relative price of investment goods. We add to this classical channel several additional ones: a decrease in mortality rates, a decline in the labor share, and changes in government debt.

Our model is able to explain the decrease in real interest rates over the past forty years. The reduction in fertility, mortality, and the rate of productivity growth play the largest role in the decrease in real interest rates. The main factor which has tended to counterbalance these forces is an increase in government debt over this time period. Changes in the labor share and in the relative price of investment goods play a smaller role in explaining the decline in real interest rates.

We also evaluate quantitatively under what assumptions one should expect real interest rate to return to a more ”normal” steady state of 1 percent real, the maintained assumption by the Federal Reserve projections. At a 1 percent steady state real interest rate the ZLB is much less likely to pose a problem for business cycle stabilization (see, for example, Williams (2016)). A key condition under which this is to be expected is if productivity growth increases to over 2 percent, closer to what was seen prior to the slowdown in the 1970's. This simulation make clear that the lively debate between Robert Gordon and others about the likely future evolution of productivity is crucial in determining if we should secular stagnation to remains a problem. Gordon (2015), famously, takes a very pessimistic view about the evolution of future productivity, while Brynjolfsson and McAfee (2014) takes a more optimistic view. Our simulation suggest the stakes are high in that debate for the future conduct of monetary policy, as it may determined the extend to which the ZLB remains a problem for macroeconomic management.

The paper is organized as follows. In Section 2, we start with an endowment economy to focus on interest rate determination in an OLG model with no nominal frictions. We show how debt deleveraging modeled as a tightening of collateral constraints can lower the natural rate of interest and show how slower productivity growth, slower population growth, and increased income inequality can also lower interest rates.

In Sections 3 and 4, we consider prices and inflation and incorporate nominal rigidities. We first show, that with a nominal good and the zero lower bound, a negative natural rate of interest places a bound on the steady state rate of inflation. If the steady state real interest rate must be -4%, the inflation in steady state must be at least 4% or higher in order for the zero lower bound not to be violated. We incorporate nominal frictions via downward nominal wage rigidity. Households supply a constant inelastic level of labor but nominal wages are subject to a wage norm - workers expect their nominal wages to grow at a constant rate. If the natural rate of interest falls too low, real interest rates are too high and demand falls below supply. This triggers a fall in inflation causing the wage norm to bind and raising real wages. The good market clears now, not because the real interest falls, but because output falls and real wages rise. Labor demand falls below labor supply and some fraction of the labor force faces involuntary unemployment.

While we adopt a particular nominal friction here to close the model, the basic mechanism at work in a secular stagnation is present with other types of nominal rigidities. We show in the appendix that a similar secular stagnation equilibrium emerges under standard Calvo pricing. We also show in Section 5 that a secular stagnation can emerge if hysteresis mechanisms take hold, which would be consistent with nominal frictions only playing a temporary role. A period of elevated unemployment can, for example, cause scarring of the labor force reducing potential output. Under this adjustment mechanism, the rise in unemployment and fall in inflation below target is temporary, but output does not return to its pre-trend level and interest rates remain stuck at the zero lower bound.

In Section 6, we consider both monetary and fiscal policy. We show how an increase in the inflation target results in multiple steady states, and we show that these steady state are both locally determinate. In other words, for small shock, their is a unique equilibrium path for the economy back to this local steady state. In particular, this implies the possibility of secular stagnation business cycles. In our model, small, transitory shocks could result in fluctuations in output and inflation around a long-run depressed steady state. We also establish here how fiscal policy - both increases in government spending and increases in public debt - operate in our model

In Section 7, we outline a 56- generation lifecycle model with capital accumulation and borrowing constraints. Our model incorporates a bequest motive and households face mortality risk in each period. Here, production is now a general CES with capital and labor as inputs. We incorporate both monopolistically competitive retailers and an exogenous price of capital goods that can change over time. We calibrate this model using standard moments and data of demographics and productivity growth.

2 Endowment Economy

We start by considering a simple overlapping generations economy to analyze the determination of interest rates in a OLG setting. Households live for three periods. They are born in period 1 (young), enter middle age in period 2 (middle age), and retire in period 3 (old). Consider the case in which no aggregate savings is feasible (i.e. there is no capital), but that generations can borrow and lend to one another. Moreover, imagine that only the middle and old generation receive any income in the form of an endowment: Y_t^m and Y_t^o. In this case, the young will borrow from the middle-aged households, which in turn will save for retirement when they are old and fully consume their remaining income and assets. We assume, however, that there is a limit on the amount of debt the young can borrow. Generally, we would like to think of this as reflecting some sort of incentive constraint, but for the purposes of this paper, it will simply take the form of an exogenous constant D_t (as for the "debtors" in Eggertsson and Krugman (2012)).

More concretely, consider a representative household of a generation born at time t. It has the following utility function:

$$\max_{C_t^y, C_{t+1}^m, C_{t+2}^o} \mathbb{E}_t \left\{ \log\left(C_t^y\right) + \beta \log\left(C_{t+1}^m\right) + \beta^2 \log\left(C_{t+2}^o\right) \right\}$$

where C_t^y is the consumption of the household when young, C_{t+1}^m its consumption when middle aged, and C_{t+2}^o its consumption while old. We assume that borrowing and lending take place via a one period riskfree bond denoted B_t^i where $i = y, m, o$ at an interest rate r_t. Given this structure, we can write the budget constraints facing households of the generation born at time t in each period as:

$$C_t^y = B_t^y \tag{1}$$

$$C_{t+1}^m = Y_{t+1}^m - (1+r_t)\,B_t^y + B_{t+1}^m \tag{2}$$

$$C_{t+2}^o = Y_{t+2}^o - (1+r_{t+1})\,B_{t+1}^m \tag{3}$$

$$(1+r_t)B_t^i \leq D_t \tag{4}$$

where equation (1) corresponds to the budget constraint for the young where consumption is constrained by what the household can borrow. Equation (2) corresponds to the budget constraint of the middle-aged household that receives the endowment Y_t^m and repays what was borrowed when young as well as accumulating some additional savings B_{t+1}^m for retirement. Finally, equation (3) corresponds to the budget constraint when the household is old and consumes its savings and endowment received in the last period.[2]

The inequality (4) corresponds to the exogenous borrowing limit (as in Eggertsson and Krugman (2012)), which we assume binds for the young so that[3]:

$$C_t^y = B_t^y = \frac{D_t}{1+r_t} \tag{5}$$

where the amount of debt that the young can borrow depends on their ability to repay when middle aged and, therefore, includes interest payments (hence a drop in the real interest rate increases borrowing by the young).

The old at any time t will consume all their income so that:

$$C_t^o = Y_t^o - (1+r_{t-1})\,B_{t-1}^m \tag{6}$$

The middle aged, however, are at an interior solution and their consumption-saving choices satisfies an Euler equation given by:

$$\frac{1}{C_t^m} = \beta \mathbb{E}_t \frac{1+r_t}{C_{t+1}^o} \tag{7}$$

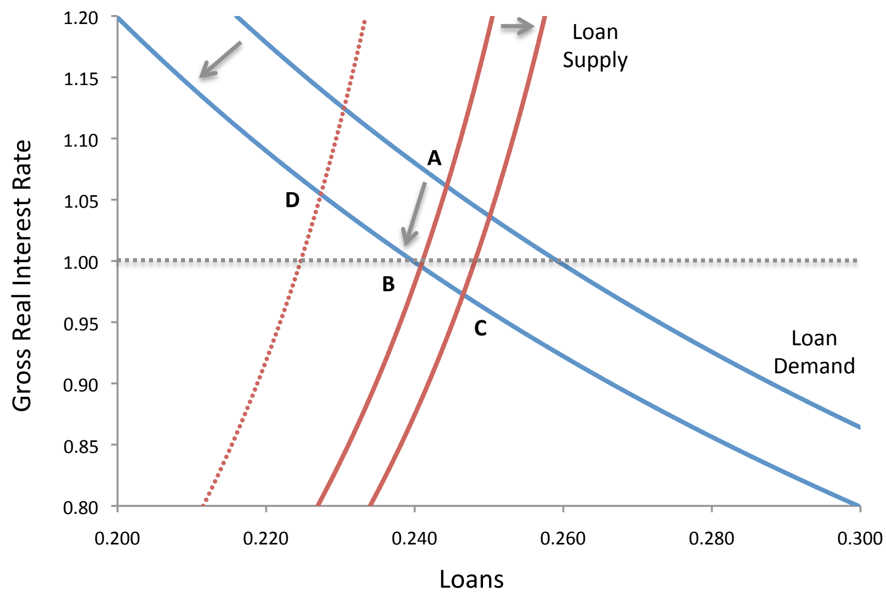
We assume that the size of each generation is given by N_t. Let us define the growth rate of the new cohort by $1+g_t = \frac{N_t}{N_{t-1}}$. Equilibrium in the bond market requires that borrowing of the young equals the savings of the middle aged so that $N_t B_t^y = -N_{t-1} B_t^m$ or:

$$(1+g_t)B_t^y = -B_t^m \tag{8}$$

[2]The addition of a warm glow bequest motive would not affect our qualitative results as the steady state real interest may still remain negative. Thwaites (2015) shows that bequests do not materially impact determination of real interest rates in an OLG setting. Nevertheless, in the full lifecycle model, we will include an explicit bequest motive.

[3]For this variation of the model the condition for this to be the case is $\frac{Y_t^m + Y_{t+1}^o/(1+r_t)}{1+\beta(1+\beta)} > D_{t-1}$. We check in our numerical experiments that the relevant condition is satisfied.

Figure 1: Equilibrium in the asset market



An equilibrium is now defined as a set of stochastic processes $\{C_t^y, C_{t,}^o, C_t^m, r_t, B_t^y, B_t^m\}$ that solve $(1), (2), (5), (6), (7)$, and (8) given an exogenous process for $\{D_t, g_t\}$.

To analyze equilibrium determination, let us focus on equilibrium in the market for savings and loans given by equation (8) using the notation L_t^s and L_t^d; the left-hand side of (8) denotes the demand for loans, L_t^d, and the right-hand side its supply, L_t^s. Hence the demand for loans (using (5)) can be written as:

$$L_t^d = \frac{1 + g_t}{1 + r_t} D_t \tag{9}$$

while the supply for savings - assuming perfect foresight for now - can be solved for by substituting out for C_t^m in (2), using (3) and (7), and for B_t^y by using (5). Then solving for B_t^m, we obtain the supply of loans given by:

$$L_t^s = \frac{\beta}{1+\beta} \left(Y_t^m - D_{t-1}\right) - \frac{1}{1+\beta} \frac{Y_{t+1}^o}{1+r_t} \tag{10}$$

Asset market equilibrium, depicted in Figure 1, is then determined by the intersection of the demand (L_t^d) and supply (L_t^s) for loans at the equilibrium level of real interest rates given by:

$$1 + r_t = \frac{1+\beta}{\beta} \frac{(1+g_t)\, D_t}{Y_t^m - D_{t-1}} + \frac{1}{\beta} \frac{Y_{t+1}^o}{Y_t^m - D_{t-1}} \tag{11}$$

Observe that the real interest rate will in general depend on the relative income between the middle aged and the young as well as on the debt limit, population growth, and the discount factor.

2.1 Productivity, Population Growth and Inequality

In an OLG setting, any factor that affects the relative supply or demand for loans has affects on the interest rate. Unlike in the standard representative agent model used in business cycle models, these forces can have permanent effect on the interest rate, and we should expect these dynamics to play out over an extended period of time. Below we offer a few examples of these forces; the list is far from exhaustive, and in the quantitative section we take a firmer stand on which of these forces best account for declining US interest rates.

Let us first consider a force that has been commonly associated with discussion on secular stagnation: a fall in total factor productivity growth (this hypothesis is most forcefully articulated by Gordon (2015)). More concretely assume that the income of the middle aged and the old is proportional to the aggregate endowment Y_t, which in turn is proportional to productivity so that $Y_t = A_t\bar{Y}$. Moreover, as the debt limit presumably reflect the extent to which the middle age agents can replay their debt, we assume that it grows with the middle aged income so that the debt limit relevant to the young at time t is given by $D_t = A_{t+1}\bar{D}$. The interest rate can then be expressed in terms of renormalized variables $\tilde{Y}_t^i$ where $i = m, o$ and $\tilde{D}_t$ were we divide by productivity:

$$1 + r_t = \frac{1+\beta}{\beta}\frac{(1+g_t)\,\tilde{D}_t\frac{A_t}{A_{t-1}}}{\tilde{Y}_t^m - \tilde{D}_{t-1}} + \frac{1}{\beta}\frac{\tilde{Y}_{t+1}^o\frac{A_{t+1}}{A_t}}{\tilde{Y}_t^m - \tilde{D}_{t-1}}$$

Consider now the effect of a slowdown in productivity. First, it shift out the supply for savings because of expectations of lower future income Y_{t+1}^o induce the middle generation to increase saving for retirement. Second, the expectation of lower future productivity tightens the borrowing constraint of the young, leading to a backward shift in the demand of loans. The new equilibrium is shown in point C in Figure 1. A key difference in how productivity affect the interest rate here relative to the representative agent, is that works both on the demand and the supply side of savings. More importantly, unlike in the representative agent model, it is not required for productivity to be falling over time in order for the real interest rate to become negative.[4] Instead, interest rates depend on how income is distributed over the lifecycle.

The mechanism by which a reduction in population growth lowers the interest rate is straightforward and can be seen directly by inspecting the expression for loan demand L_t^d. As the number of young decreases relative to the middle age (a lower g_t), this leads to a reduction in loan demand, thus shifting back the L_t^d curve and lowering the real interest rate at point B in Figure 1. In the more general quantitative model we will also consider two additional mechanism whose

[4]In the representative household model (without population growth) the real interest rate is given by:

$$1 + r_t = \beta^{-1}\frac{A_{t+1}}{A_t}$$

so productivity would need to decline at a rate greater than the inverse of the discount factor for the interest rate to be negative.

influence should be relatively intuitive. We allow for the possibility of decreases in mortality and for a bequest motive. Both these forces work towards increasing the supply of savings similarly depressing the interest rate. The former effect is particularly powerful quantitatively in explaining the decline in interest rates observed in the US over the past decades.

Our model can also be used to consider conditions under which rising inequality could lower interest rates. There is no general result about how an increase in inequality should affect the real rate of interest in our model. In general, this relationship will depend on how changes in income affect the relative supply and demand for loans. There are plausible conditions, however, under which higher inequality will in fact reduce the natural rate of interest. We provide one such example in the appendix, in which a fraction of the middle age population are credit constrained. In this case shifting income from the low skilled, credit constrained households to the high skilled in middle age reduces the real interest rate. In general, the condition needed for inequality to have an effect, is simply that those with higher income at a given age level, save more than those with lower income.[5]

Lastly, a fourth often identified with secular stagnation is a fall in the relative price of investment goods. As investment goods become cheaper, each dollar of saving is able to buy the same amount of capital, thereby reducing total nominal investment demand. Since our illustrative example does not include capital, we will defer this discussion to Section 7 where we introduce capital in the full lifecycle model.

2.2 Deleveraging

One way to understand the secular stagnation hypothesis is that longer term forces like demographics account for the downward trend in interest rates, but the housing crisis and subsequent consumer deleveraging pushed the natural rate of interest well below zero. The effects of debt deleveraging operates in our framework is different from, for example, Eggertsson and Krugman (2012). In that model and other models of consumer deleveraging, interest rates fall on impact but then rise once the deleveraging cycle is complete as debt converges to a lower level. In our framework, there is no deleveraging cycle - interest rates do not naturally recover over time.

Point B in Figure 1 shows the equilibrium level of the real interest rate on impact after a deleveraging shock. As we can see, the shock leads directly to a reduction in the demand for loans since the demand curve shifts from $\frac{1+g_t}{1+r_t}D^H$ to $\frac{1+g_t}{1+r_t}D^L$. The supply of loanable funds is un-

[5] Most realistic models of bequests, for example, argue that the preference for leaving a bequest increase with income implying increases in bequests with higher inequality. See Dynan, Skinner and Zeldes (2004) for discussion of other mechanisms, such as persistence of skill types across generations. We have also experimented with a production structure where skill-bias technological change (as in Krusell et al. (2000)) increases inequality and found plausible condition under which this process puts downward pressure on the real interest rate. We leave a fuller analysis of this mechanism for future research.

changed as can be seen in (10) since the debt repayment of the middle generation depends upon the lagged collateral constraint D_{t-1}.

Let us compare the equilibrium in point B to point A. Relative to the previous equilibrium, the young are now spending less at a given interest rate, while the middle aged and old are spending the same. Since the endowment must be fully consumed in our economy, this fall in spending by the young then needs to be made up by inducing some agents to spend more. This adjustment takes place via reduction in the real interest rate. The drop in the real interest stimulates spending via two channels. As equation (11) shows, a fall in the real interest rate makes consumption today more attractive to the middle aged, thus increasing their spending.[6] For the credit-constrained young generation, a reduction in the real interest rate relaxes their borrowing constraint. A lower interest rate allows them to take on more debt, B_t^y at any given D_t because their borrowing is limited by their ability to repay in the next period and that payment depends on the interest rate. Observe that the spending of the old is unaffected by the real interest rate at time t; these households will simply spend all their existing savings and their endowment irrespective of the interest rate.

So far, the mechanism described in our model is exactly the same as in Eggertsson and Krugman (2012) and the literature on deleveraging. There is a deleveraging shock that triggers a drop in spending by borrowers at the existing rate of interest. The real interest rate then needs to drop for the level of aggregate spending to remain the same. In Eggertsson and Krugman (2012), the economy reaches a new steady state next period in which, once again, the real interest rate is determined by the discount factor of the representative savers in the economy. In that setting, the loan supply curve shifts back so that the real interest rate is exactly the same as before (as seen in point D). Loan supply shifts back in Eggertsson and Krugman (2012) because borrower deleveraging reduces interest income accruing to savers which implies that their supply of savings falls in equilibrium.

In contrast, in our model there is no representative saver; instead, households are both borrowers and savers at different stages in their lives. The fall in the borrowing of the young households in period t then implies that in the next period - when that agent becomes a saver household - the middle-aged agent has more resources to save (since he has less debt to pay back due to the reduction in D_t). Therefore, at time $t+1$ the supply of savings L_t^s shifts outwards as shown in Figure 1. In sharp contrast to Eggertsson and Krugman (2012), where the economy settles back on the old steady state after a brief transition with a negative interest rate during the deleveraging

[6] Observe that the L_t^s curve is upward sloping in $1+r_t$. A drop in the real interest rate reduces desired saving as the present value of future income rises. If preferences exhibit very weak substitution effects and if the endowment is received only by the middle-aged generation, it is possible to have a downward sloping L_t^s curve. Saving is increasing in the interest rate (and conversely demand that increases when interest rates fall) is, in our view, the empirically relevant case.

phase, the economy settles down at a new steady state with a permanently lower real rate of interest that may be negative depending on the size of the shock. This process can serve as a powerful and persistent propagation mechanism for the original deleveraging shock.

3 Price Level Determination

With nominal price determination, it becomes clear that if the real interest rate falls negative permanently negative, there is no equilibrium consistent with inflation below a certain level. For example, if the real interest rate is -3 percent, then inflation needs to be higher than 3 percent. This will have fundamental implications when we introduce realistic nominal frictions; the unwillingness or inability of the central bank to accommodate high enough inflation rate will result in permanent contraction in output.

As is by now standard in the literature, we introduce a nominal price level by assuming that one period nominal debt denominated in money is traded, and that the government controls the rate of return of this asset (the nominal interest rate).[7] The saver in the previous section (middle-generation household) now has access to risk-free nominal debt which is indexed in dollars in addition to one period risk-free real debt.[8] This assumption gives rise to the consumption Euler equation which is the nominal analog of equation (7):

$$\frac{1}{C_t^m} = \beta E_t \frac{1}{C_{t+1}^o}(1+i_t)\frac{P_t}{P_{t+1}} \tag{12}$$

where i_t is the nominal rate and P_t is the price level. We impose a non-negativity constraint on nominal rates. Implicitly, we assume that the existence of money precludes the possibility of a negative nominal rate. At all times:

$$i_t \geq 0 \tag{13}$$

Equation (7) and (12)imply (assuming perfect foresight) the standard Fisher relation:

$$1 + r_t = (1+i_t)\frac{P_t}{P_{t+1}} \tag{14}$$

where again r_t is exogenously determined as before by equation (11). The Fisher equation simply states that the real interest rate should be equal to the nominal rate deflated by the growth rate of the price level (inflation).[9]

[7]There are various approaches to microfound a demand for money by using money in the utility function or cash-in-advance constraints.

[8]For simplicity, we assume that this asset trades in zero net supply, so that in equilibrium the budget constraints already analyzed are unchanged. However, we relax this assumption once we incorporate fiscal policy

[9]Again, we can define an equilibrium as a collection of stochastic processes $\{C_t^y, C_{t,}^o, C_t^m, r_t, i_t, B_t^y, B_t^m, P_t\}$ that solve (1), (2), (5), (6), (7) and (8) and now in addition (12) and (13) given the exogenous process for $\{D_t, g_t\}$ and some policy reaction function for the monetary authority like an interest rate rule.

From (13) and (14), it should be clear that if the real rate of interest is permanently negative, there is no equilibrium consistent with stable prices. To see this, assume there is such an equilibrium so that $P_{t+1} = P_t = P^*$. Then the Fisher equation implies that $i_t = r_t < 0$ which violates the zero bound. Hence a constant price level - price stability - cannot be sustained when r_t is negative.

Let us denote the growth rate of the price level (inflation) by $\Pi_t = \frac{P_{t+1}}{P_t} = \bar{\Pi}$. The zero bound and the Fisher equation then implies that for an equilibrium with constant inflation to satisfy the ZLB, there is a bound on the inflation rate given by $\Pi(1 + r) = 1 + i \geq 1$ or:

$$\bar{\Pi} \geq \frac{1}{1+r} \tag{15}$$

which implies that steady state inflation is bounded from below by the real interest rate due to the zero bound.

Observe that at a positive real interest rate this bound may seem of little relevance. If, as is common in the literature using representative agent models, the real interest rate in steady state is equal to inverse of the discount factor, then this bound says that $\Pi \geq \beta$. In a typical calibration this implies a bound on the steady state inflation rate of about -2% to -4%; well below the inflation target of most central banks meaning that this bound is of little empirical relevance.

With a negative real rate, however, this bound takes on a greater practical significance. If the real interest rate is falls negative, it implies that, under flexible prices, steady state inflation needs to be permanently above zero and possibly well above zero depending on the value of the steady state real interest rate. This insight will be critical once we move away from the assumption that prices are perfectly flexible. In that case, if the economy calls for a positive inflation rate - and cannot reach that level due to policy (e.g. a central bank committed to low inflation) - the consequence will be a permanent drop in output.

4 Aggregate Supply

So far, we have focused on the determination of interest rates and level of inflation required to satisfy negative natural rates at the zero lower bound. In this section, we incorporate nominal rigidities and examine output and inflation adjustment in the presence of nominal rigidities and a monetary policy rule that maintains an inflation target that is too low. Our message is that with plausible nominal rigidities and the ZLB, adjustment from a natural rate of interest that falls negative may come from a reduction in output rather than an increase in the inflation rate. While we model nominal frictions in a particular way - as downward nominal wage rigidity - we further argue that the qualitative features of secular stagnation would survive with alternative nominal frictions or in the presence of hysteresis.

Before specifying the particular way we incorporate wage rigidity, a broader context is warranted. There seems to be a relatively broad professional consensus among economists, that at high inflation, then expectation will adjust so that there is no output effect in the long run, an empirical prediction born out in the 1970's in the US. This viewpoint is often summarized by a vertical aggregate supply curve in the long run (see Figure 2). However, such consensus has never been as strong with respect to the neutrality of low inflation or deflation. This viewpoint was for example summarized by Tobin (1972) who pointed out that during the Great Depression there was a strong reluctance on the part of firms to cut back nominal wages despite high unemployment. This, Tobin suggested, should lead to a permanent tradeoff between inflation and output at low inflation or when there is deflation as seen in Figure 2.

From the perspective of our theory of secular stagnation, a long-run tradeoff between inflation and output is all that is need to ensure the existence of a stagnation equilibrium. There are several way in which such trade-off can be generated, for example in the standard New Keynesian Calvo model, this trade-off appear due to the presence of price dispersion (see Appendix D). Here, we opt for capturing this tradeoff by introducing downward nominal wage rigidity; this specification has the virtue of capturing the neutrality of inflation at high inflation rates, as in the 1970's, yet simultaneously giving rise to meaningful tradeoffs at low inflation rates as observed today in the industrial world as well as during the Great Depression.

There is a relatively large empirical literature that documents the prevalence of downwardly rigid wages, even in the face of high unemployment. Bewley (1999) who interviewed a number of firms and documented a reluctance to cut nominal wages. The presence of substantial nominal wage rigidity has been established empirically recently in US administrative data by Fallick, Lettau and Wascher (2011), in worker surveys by Barattieri, Basu and Gottschalk (2010), and in cross-country data by Schmitt-Grohé and Uribe (2015). Our model specification is closely related to the supply side specification of Schmitt-Grohé and Uribe (2015), and we rely on their estimates for the degree of wage rigidity in our quantitative analysis in Section 7.[10] Relative to their specification, our specification allows us to vary the degree to which wages are downwardly rigid or flexible and show that increasing flexibility does not lead to stabilization, suggesting a basic failure of the price mechanism to achieve full employment in a secular stagnation.

We simplify our exposition by assuming that only the middle generation receive income. We assume that this generation supplies labor inelastically. The budget constraint of the agents is again given by equations (1) and (4), but now we replace the budget constraint of the middle generation (2) and old generation (3) with:

$$C^m_{t+1} = \frac{W_{t+1}}{P_{t+1}} L_{t+1} + \frac{Z_{t+1}}{P_{t+1}} - (1+r_t) B^y_t + B^m_{t+1} \tag{16}$$

$$C^o_{t+2} = -(1+r_{t+1}) B^m_{t+1} \tag{17}$$

[10] Also see Shimer (2012) and Kocherlakota (2013) for further discussion of how wage rigidities can explain labor dynamics the Great Recession.

where W_{t+1} is the nominal wage rate, P_{t+1} the aggregate price level, L_{t+1} the labor supply of the middle generation, and Z_{t+1} the profits of the firms. For simplicity, we assume that the middle generation will supply its labor inelastically at $\bar{L}$. Note that if the firms do not hire all available labor supplied, then labor demand L_t may be lower than $\bar{L}$ due to rationing. Under these assumption, each of the generations' consumption-saving decisions remain the same as before.

On the firm side, we assume that firms are perfectly competitive and take prices as given. They hire labor to maximize period-by-period profits. Their problem is given by:

$$Z_t = \max_{L_t} P_t Y_t - W_t L_t \tag{18}$$

$$\text{s.t.} \quad Y_t = L_t^{\alpha} \tag{19}$$

The firms' labor demand condition is then given by:

$$\frac{W_t}{P_t} = \alpha L_t^{\alpha-1} \tag{20}$$

So far we have described a perfectly frictionless production side, and, if this were the end, our model would be analogous to what we have already considered in the endowment economy. Output would now be given by $Y_t = L_t^{\alpha} = \bar{L}^{\alpha}$ and equation (20) would determine the real wage.

However, consider a world in which households will never accept working for wages that fall below their wage in the previous period so that nominal wages at time t cannot be lower than what they were at time $t-1$. Or slightly more generally, imagine that the household would never accept lower wages than a wage norm given by $\tilde{W}_t = \gamma W_{t-1} + (1-\gamma) P_t \alpha \bar{L}^{\alpha-1}$. If $\gamma = 1$, the wage norm is last period nominal wages so wages are perfectly downwardly rigid, but, if $\gamma = 0$, we obtain the flexible price nominal wage considered earlier. Formally, if we take this assumption as given, it implies that we replace $L_t = \bar{L}$ with:

$$W_t = \max\left\{\tilde{W}_t, P_t \alpha \bar{L}^{\alpha-1}\right\} \text{ where } \tilde{W}_t = \gamma W_{t-1} + (1-\gamma) P_t \alpha \bar{L}^{\alpha-1} \tag{21}$$

We see that nominal wages can never fall below the wage norm $\tilde{W}_t$ with γ parameterizing the degree of rigidity. If labor market clearing requires higher nominal wages than the past nominal wage rate, this specification implies that labor demand equals labor supply and the real wage is given by (20) evaluated at $L_t = \bar{L}$.

To close the model, we must specify a monetary policy rule. Suppose that the central bank sets the nominal rate according to a standard Taylor rule:

$$1 + i_t = \max\left(1, (1+i^*)\left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi}\right) \tag{22}$$

where $\phi_\pi > 1$ and Π^* and i^* are parameters of the policy rule that we hold constant. This rule states that the central bank tries to stabilize inflation around an inflation target Π^* (determined by the steady state nominal interest rate i^* and the inflation target Π^*) unless it is constrained by the zero bound.[11]

Definition 1. *A competitive equilibrium is a sequence of quantities* $\{C_t^y, C_{t,}^o, C_t^m, B_t^y, B_t^m, L_t, Y_t, Z_t\}$ *and prices* $\{P_t, W_t, r_t, i_t\}$ *that satisfy* (1), (5), (6), (7), (8), (12), (13), (16), (18), (19), (20), (21) *and* (22) *given an exogenous process for* $\{D_t, g_t\}$ *and initial values for* W_{-1} *and* B_{-1}^m.

Having defined a competitive equilibrium, we can now analyze the steady state of the model. We can graphically represent the steady state of our model using two equations that relate output and inflation. Effectively, we can combine the relationship between output and the real interest rate derived in Section 2 with the Fisher relation and monetary policy rule to obtain an aggregate demand curve. Analogously, by combining the wage norm, production function, and the labor demand condition, we can obtain an aggregate supply curve. The intersection of these curves determines output and inflation in steady state.

The aggregate supply specification of the model consists of two regimes: one in which real wages equal the market clearing real wage (if $\Pi \geq 1$), and the other when the bound on nominal wages is binding ($\Pi < 1$). Intuitively, positive inflation in steady state means that wages behave as if they are flexible since nominal wages must rise to keep real wages constant.[12] If $\Pi \geq 1$, then labor demand equals the exogenous level of labor supply $\bar{L}$ defining the full employment level of output Y^f:

$$Y = \bar{L}^\alpha = Y^f \text{ for } \Pi \geq 1 \tag{23}$$

This is shown as a solid vertical segment in the AS curve in Figure 2.

When the inflation rate in steady state is negative - $\Pi < 1$, the wage norm binds and real wages exceed the market-clearing real wage. For $\Pi < 1$, we can derive a relationship between output and inflation by using equation (19) and equation (20) to eliminate real wages and labor; we obtain the following AS curve:

$$\frac{\gamma}{\Pi} = 1 - (1-\gamma)\left(\frac{Y}{Y^f}\right)^{\frac{1-\alpha}{\alpha}} \quad \text{for } \Pi < 1 \tag{24}$$

Equation (24) is simply a nonlinear Phillips curve. The intuition is straightforward: as inflation increases, real wages fall and firms hire more labor. As γ approaches unity, the Phillips curve flattens, and as γ approaches zero, the Phillips curve becomes vertical. Importantly, this Phillips curve relationship is not a short-run relationship; instead, it describes the behavior of steady state

[11]In the Appendix E, we introduce money explicitly into our model and show that our results are not affected so long as fiscal policy keeps consolidated government liabilities stable.

[12]First observe that for W to exceed $\tilde{W}$ in steady state, it must be the case that the real wage - denoted $w = \frac{W}{P}$) - is $w \geq \gamma w \Pi^{-1} + (1-\gamma)\alpha \bar{L}^{\alpha-1}$. This is satisfied as long as $\Pi \geq 1$.

inflation and output. The aggregate supply curve is shown in Figure 2, with the vertical segment given by (23) and the upward-sloping segment given by (24) with the kink at $\Pi = 1$.[13]

Turning to the aggregate demand relation, we again have two regimes (like aggregate supply): one when the zero bound is not binding, the other when it is binding. Let us start with deriving aggregate demand at positive nominal rates. Combining the interest rate expression, Fisher relation, and monetary policy rule - equations (11) , (14) , (22) and assuming $i > 0$ - we get:

$$Y = D + \frac{(1+\beta)(1+g)D\Gamma^*}{\beta}\frac{1}{\Pi^{\phi_\pi - 1}} \text{ for } i > 0 \tag{25}$$

where $\Gamma^* \equiv (1+i^*)^{-1}(\Pi^*)^{\phi_\pi}$ is the composite policy parameter given in the reaction function. The upper portion of the AD curve in Figure 2 depicts this relationship. As inflation increases, the central bank raises the nominal interest rate by more than one for one (since $\phi_\pi > 1$), which, in turn, increases the real interest rate and reduces output demand.

At the zero lower bound, we combine the same set of equation and now set $i = 0$. We obtain the following expression relating output and inflation:

$$Y = D + \frac{(1+\beta)(1+g)D}{\beta}\Pi \text{ for } i = 0 \tag{26}$$

Now the AD curve is upward sloping. The logic should again be relatively straightforward for those familiar with the literature on the short-run liquidity trap: as inflation increases, the nominal interest rate remains constant, thus reducing the real interest rate. This change in the real rate raises consumption demand as shown by the bottom portion of the AD curve in Figure 2.

The kink in the aggregate demand curve occurs at the inflation rate at which monetary policy is constrained by the zero lower bound. That is, the AD curve depicted in Figure 2 will become upward sloping when the inflation rate is sufficiently low that the implied nominal rate the central bank would like to set is below zero. Mathematically, we can derive an expression for this kink point by solving for the inflation rate that equalizes the two arguments in the max operator of equation (22):

$$\Pi_{kink} = \left(\frac{1}{1+i^*}\right)^{\frac{1}{\phi_\pi}}\Pi^* \tag{27}$$

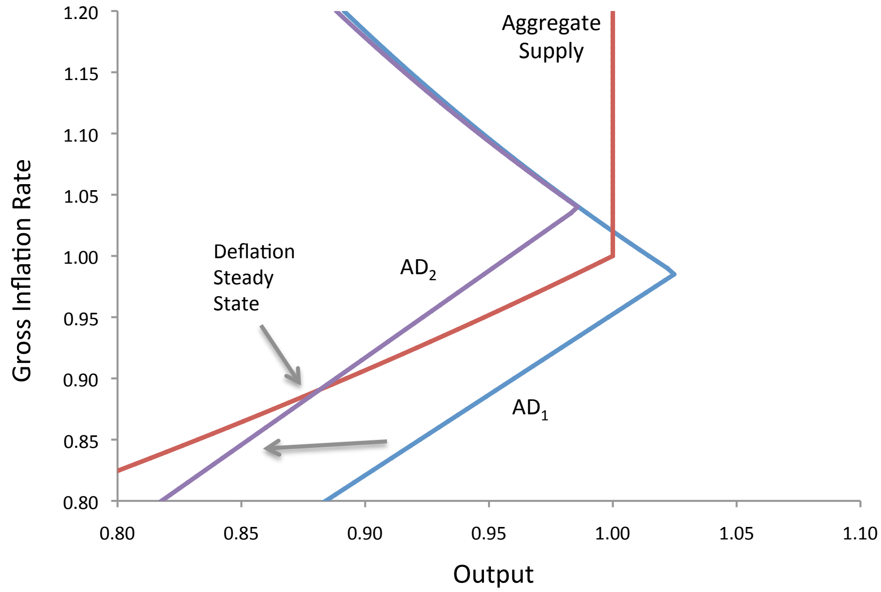
The location of the kink in the AD curve depends on both parameters of the policy rule: the inflation target Π^* and the targeted real interest rate i^*.

In what follows, it will be useful to define the natural rate of interest - the interest rate at which output is at its full employment level. The natural rate can be obtained by evaluating equation (11) at the full employment level of output Y^f:

$$1 + r_t^f = \frac{1+\beta}{\beta}\frac{(1+g_t)D_t}{Y^f - D_{t-1}}$$

[13] The AS and AD diagrams and numerical examples in this section assume the following parameter values: $\beta = 0.985$, $\gamma = 0.94$, $\alpha = 0.7$, $\Pi^* = 1.01$, $\phi_\pi = 2$, $D = 0.28$, $g = 0.9\%$.

Figure 2: Steady state aggregate demand and aggregate supply curves



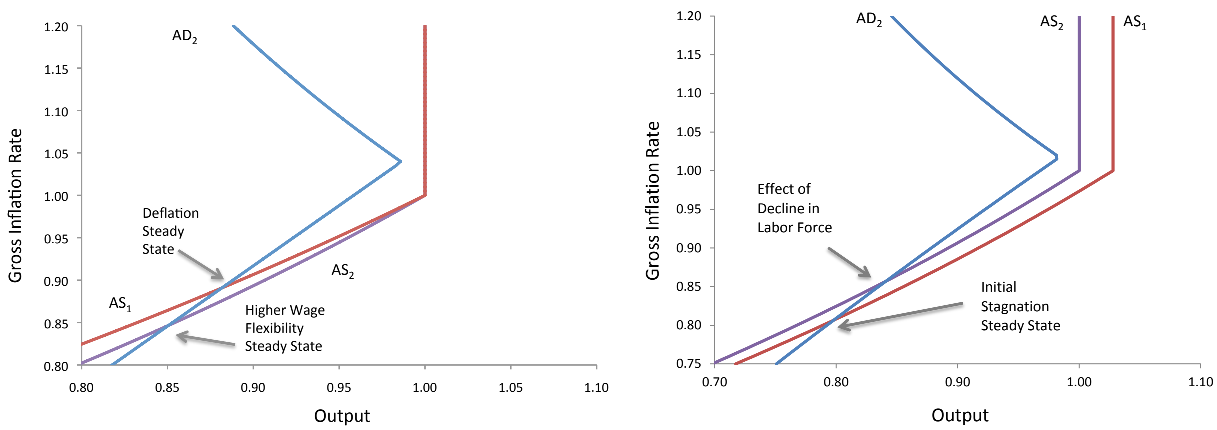
It is useful to note that the full employment interest rate corresponds exactly to the real interest rate we derived in the endowment economy. Hence any of the forces we have already shown to have reduce the real interest rate in the endowment economy, will directly affect the full employment real interest rate in the more general setup.

5 Full Employment and Secular Stagnation

As we will show, the AD curve can intersect with the AS curve in either the full-employment region (vertical segment of the AS curve) or in the secular stagnation region (upward sloping segment of the AS curve). Consider first a full employment equilibrium when the natural rate of interest is positive. Here, we assume that the central bank aims for a positive inflation target (that is, $\Pi^* > 1$) and assume that the nominal interest rate is consistent with the Taylor rule: $1 + i^* = (1 + r^f)\Pi^*$ With $r_f > 0$, then the aggregate demand curve intersects the aggregate supply curve on the vertical segment of the AS curve. The exact intersection point is determined by the inflation target.[14] This full employment equilibrium is displayed in Figure 2. Importantly, if the economy is impacted by forces that reduce the natural rate of interest, the intersection point is unchanged so long as $r_f > 0$. The central bank will fully offset these shocks via reductions in the nominal interest rate. Under our assumed policy rule, the equilibrium depicted in Figure 2 is

[14] In case $\Pi^* = 1$ the intersection is at the kink of the AS curve.

Figure 3: Adjustment mechanisms: wage flexibility and hysteresis



unique for a small enough inflation target and high enough γ.[15]

The making of a secular stagnation equilibrium is shown in Figure 2. Here, we illustrate the effect of a fall in the natural rate so that $1 + r_t^f < (\Pi^*)^{-1}$. As emphasized earlier several forces may be responsible for driving the natural rate negative. For concreteness, in (25), consider a slowdown in population growth. This reduces output at any given inflation rate. This fall in output stems from the decline in consumption of the younger households due to the fact there are fewer of these borrowers relative to the middle age savers. In the normal equilibrium, this drop in spending would be compensated by a drop in the real interest rate that eases borrowing constraints and spurs the middle generation to increase consumption. However, the zero lower bound prevents this adjustment. Hence, the shock moves the economy off the full employment segment of the AS curve to a deflationary steady state where the nominal interest rate is zero. Here, steady state deflation raises steady state real wages above their market-clearing level, thus depressing demand for labor and contracting output.[16]

Proposition 1. *If $\gamma > 0$, $\Pi^* = 1$, and $i^* = r^f < 0$, then there exists a unique, locally determinate secular stagnation equilibrium.*

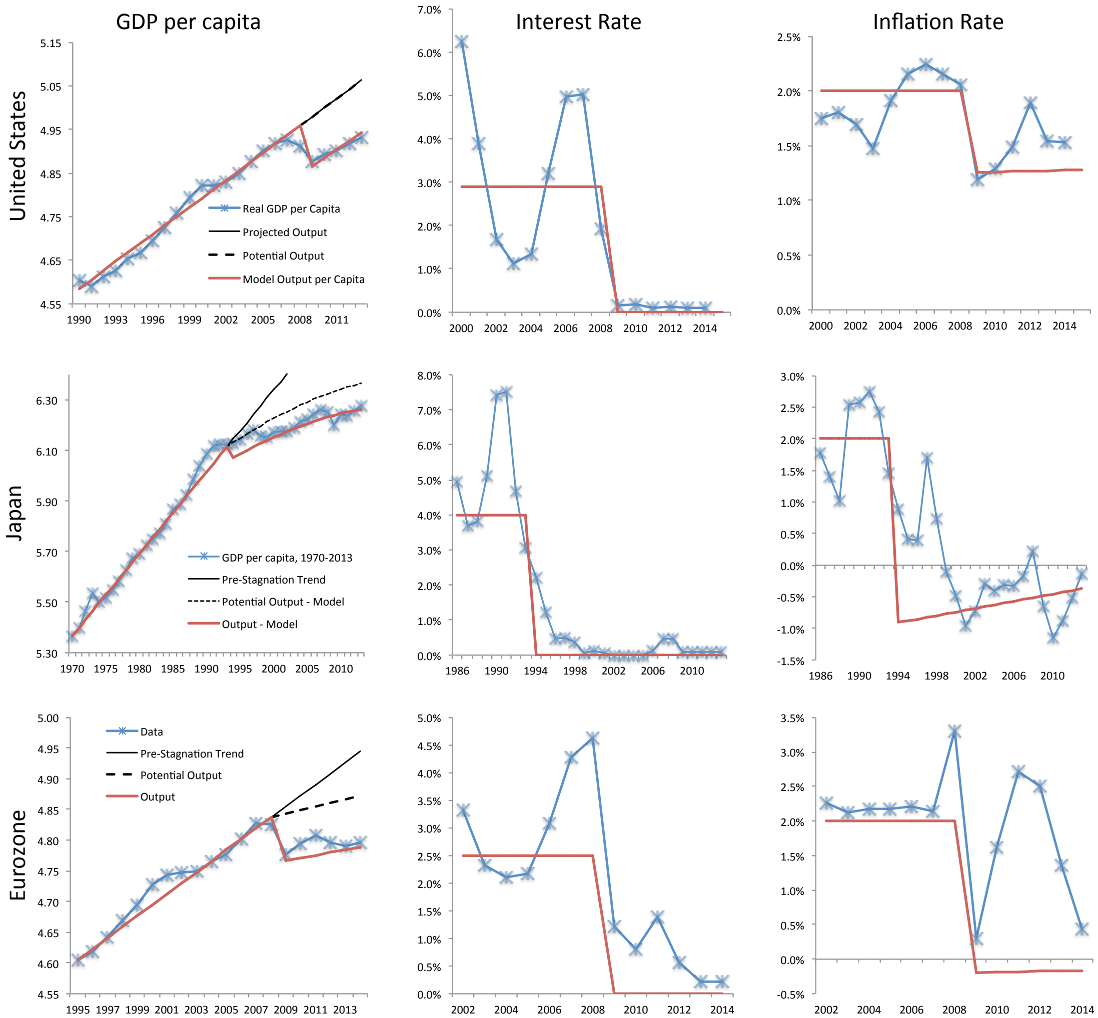
Proof. See Appendix C. □

As shown in Appendix B, if we linearize the model around the unique secular stagnation steady state, the dynamic system is locally determinate. The determinacy of the secular stagnation steady state in our model stands in contrast to the indeterminacy of the deflation steady state

[15]Uniqueness is guaranteed if $\gamma = 1$. As γ approaches zero, more equilibria are possible. We discuss these additionally equilibria when they appear in a more general setting in Section 6.

[16]As we show in the next section outright deflation is not central to the mechanism. If wages are indexed to the inflation target, then inflation that falls below target results in real wages that exceed the market-clearing level and output falls below the full-employment level.

Figure 4: Data v. model transition paths: US, Japan, Eurozone



analyzed in Schmitt-Grohé and Uribe (2013) and Benigno and Fornaro (2015).[17] Indeterminacy of equilibrium has been one justification for disregarding the deflation steady state and restricting analysis to the determinate positive interest rate steady state in New Keynesian models. As shown in Proposition 1, the AD curve must be steeper than the AS curve at the intersection point which implies the local determinacy condition. Importantly, local determinacy implies that an economy in secular stagnation may continue to experience business cycles - fluctuations in unemployment and output around a permanently depressed growth path.

Our model delivers a permanent steady state slump with no pull towards full employment. A natural question is what forces can move the economy back to full employment, absent any government intervention. First, and perhaps obviously, if the shock that pushes the natural rate negative is only temporary, then the full employment equilibrium can be attained. To the extent

[17] For further discussion on the stability properties in Benhabib, Schmitt-Grohé and Uribe (2001), see Bullard (2010).

that the shocks leading to negative real interest rates are slow moving (such as demographic factors) there is less reason to be optimistic about this adjustment mechanism.

A key friction in triggering unemployment was the fact that wages were downwardly rigid. A natural question, then, is whether an increase in the flexibility of the wage adjustment process will push the economy back to full employment. The answer to this question is no. This result emerges because an increase in price/wage flexibility triggers a drop in expected inflation, thus increasing the real interest rate. This cannot be offset by interest rate cuts due to the zero bound. As wages become more flexible (a decrease in the parameter γ), the slope of the AS curve steepens (see the left-hand panel of Figure 3). For any given deflation steady state, a decrease in γ will shift the steady state along the AD curve, increasing the rate of deflation, raising real wages, and therefore increasing the shortfall in output. This result echoes the paradox of flexibility shown in Eggertsson and Krugman (2012) and considered in a more general setting in Bhattarai, Eggertsson and Schoenle (2014).

Finally, a third adjustment mechanism is a reduction in the labor force by discouraged workers (or workers whose skills deteriorate) after prolonged spells of unemployment. Labor force participation fell markedly in the US and the duration of unemployment remains elevated in the US despite recent reductions in the unemployment rate. A reduction in the labor force reduces downward pressures on wages and weakens deflationary pressure. If the contraction of the labor force proceeds far enough, the output gap is eliminated with the AD curve intersecting the AS curve at the new, lower full employment level of output. Under hysteresis mechanisms, the AS curve shifts inwards along the transition path. Unemployment falls and inflation rises back toward target while nominal interest rates stay at the ZLB. Relative to the pre-stagnation trend, output remains depressed - in this extension, there is no catch up back to the pre-trend level of employment. Hysteresis mechanisms need not work solely through the labor market - see Garga and Singh (2016) for a model with hysteresis effects on productivity growth.

In Figure 4, we present data on the evolution of output, inflation and the short-term nominal interest rate in the US, Japan and the Eurozone. We calibrate our simple three period model and show how our model generates dynamics consistent with the observed behavior of these macro aggregates. Even though highly stylized, a straightforward calibration of our simple model does a good job of capturing the transition dynamics of output and interest rates in these stagnation episodes. To conserve space, we defer the description of the calibration underlying Figure 4 to Appendix G; this allows us to focus on a more realistic quantitative lifecycle model. The key message of the figure is that our model has, in principle, no difficulty generating a persistent fall in inflation, interest rates at the ZLB, and output persistently below trend without any return to pre-recession trend. These features are in stark contrast to the standard New Keynesian model in which the model explodes (a determinate equilibrium no longer exists) if the natural rate of interest is negative for long enough time (see Eggertsson and Singh (2016). Before proceeding to the lifecycle model, we consider implications for monetary and fiscal policy.[18]

6 Monetary and Fiscal Policy

We now turn to consideration of monetary and fiscal policy in a secular stagnation. As we show in this section, the policy implications of our model for both monetary and fiscal policy are starkly different from existing New Keynesian models of the zero lower bound. Despite increasing calls for increasing the inflation target, our model suggests that this policy response may not necessarily be effective. An increase in the inflation target only allows the *possibility* of full employment - this policy does not eliminate the secular stagnation steady state. Forward guidance is also far less useful in our model. Fiscal policy can be more effective and, since our model is non-Ricardian, changes in the public debt or forms of redistribution can also eliminate a secular stagnation. However, details of the financing of a fiscal expansion and beliefs about future fiscal policy turn out to be quite important. In other words, simply increasing government spending while the ZLB is binding - a highly expansionary policy in standard NK models - is not guaranteed to have the same effects in our framework.

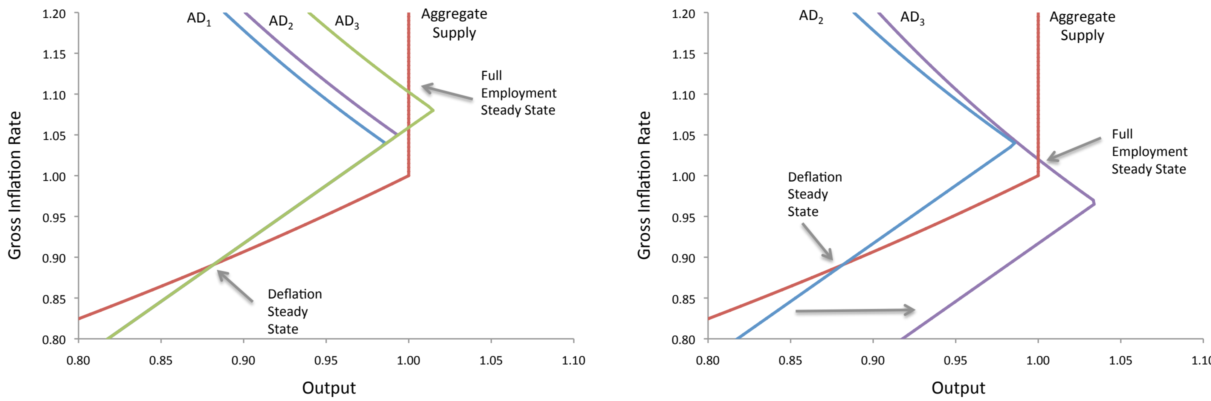
Let us first consider the effect of an increase in the inflation target Π^* in our model. This change has no effect on the AS curve but instead shifts the AD curve. Specifically, a rise in the inflation target shifts out the kink point in the AD curve as shown in the left-hand panel of Figure 5. In this figure, the initial inflation target is set at one percent. As the inflation target increases, the point at which the aggregate demand curve kinks moves upward effectively shifting up the downward sloping portion of the AD curve. AD_1 shows the original aggregate demand curve with a unique secular stagnation steady state. AD_2 shows the effect of a modest increase in the inflation target, while AD_3 shows the effect of a large increase in the inflation target. Notice that AD_3 now intersects the aggregate supply curve at three points.

AD_2 illustrates the perils of too small an increase in the inflation target. For a sufficiently negative natural rate of interest, a small increase in the inflation target will not shift the AD curve enough to intersect the full employment line. The contention that a small increase in the inflation target will be ineffective has been labeled by Krugman variously as the "timidity trap" or, in reference to Japan in the late 1990s, as the "law of the excluded middle." Our framework readily captures this idea. Formally, the inflation target needs to be high enough so that $\left(1+r^f\right)\Pi^* \geq 1$; otherwise, the kink point in the AD curve occurs to the left of the full employment line.

With a sufficiently large increase in the inflation (as shown by AD_3), our model admits three distinct steady states. The first steady state at the top intersection of the two curves is the normal,

[18] To capture the slowdown in GDP per capita growth and the absence of outright deflation in the US and Eurozone, we extend our basic model to incorporate hysteresis effects on productivity growth and a more general wage norm where nominal wages are indexed to productivity growth and the inflation target. See Appendix F.

Figure 5: Monetary and fiscal policy responses



full-employment equilibrium at which point inflation is equal to the inflation target of the central bank Π^*. At this point the nominal interest rate is positive because the inflation target is large enough to accommodate a negative shock - that is, $(1+r^f)\Pi^* > 1$. However, there is another equilibrium at full employment which is consistent with the policy rule. This is the second intersection of the two curves where $i = 0$ and $\Pi < \Pi^*$. This equilibrium, however, is locally indeterminate.[19] Most importantly, the secular stagnation steady state remains - an increase in the inflation target does not eliminate this equilibrium. Furthermore, this steady is locally determinate, meaning that small shocks have a unique path back to the secular stagnation steady state.

This multiplicity suggests that monetary policy is less effective in our environment than in models that feature temporary liquidity traps such as Krugman (1998) or Eggertsson and Woodford (2003). In those models, a permanent increase in the inflation target will always have an effect because, by assumption, one can always reach the higher inflation target at some point in the future. Working backwards, a commitment of this sort will always have expansionary effects during the liquidity trap, and, provided the inflation target is high enough, it may even eliminate the demand slump altogether. Since the trap is permanent in our model, however, this backward induction breaks down; there is no future date at which one can be certain that the higher inflation target is reached (even if the policy regime is fully credible in the sense that people do not expect the government to deviate from the policy rule). For the same reason, commitments to keep nominal rates low for a long period of time in a secular stagnation is of limited use and does not by itself guarantee a recovery. Indeed, interest rate commitments of the type currently pursued by the Federal Reserve would be irrelevant in shifting the economy out of the deflationary equilibrium since households are expecting rates to stay at zero forever. Even if a recovery were anticipated, the expansionary effects of a commitment to keep interest rates lower in the future is far less effective given discounting in the Euler equation due to finite lifetimes.[20] Though an increase in the

[19] This steady state is akin to the deflation steady state in Benhabib, Schmitt-Grohé and Uribe (2001).

[20] See McKay, Nakamura and Steinsson (2015) for a discussion.

inflation target could make a full-employment steady state feasible, our model is silent on how a government could coordinate expectations on the "good" full-employment equilibria.[21]

Given the drawbacks of monetary policy, we turn to fiscal policy. We extend our model to incorporate taxes and denote taxes on each generation by T_t^i where $i = y, m$ or o. We first consider the effect of fiscal policy on the natural rate (i.e. in the endowment economy) before reincorporating nominal frictions. The budget constraints can now be written taking taxes into account:

$$C_t^y + T_t^y = B_t^y \tag{28}$$

$$C_{t+1}^m + (1 + r_t) B_t^y = Y_{t+1}^m - T_{t+1}^m - B_{t+1}^m \tag{29}$$

$$C_{t+2}^o = Y_{t+2}^o + (1 + r_{t+1}) B_{t+1}^m - T_{t+2}^o \tag{30}$$

The key difference relative to our model without fiscal policy is that asset market clearing is now given by:

$$-N_{t-1} B_t^m = N_t B_t^y + N_{t-1} B_t^g \tag{31}$$

where B_t^g is government debt (normalized in terms of the size of the middle generation). Before, the only demand for borrowing came from the young households; now the government also may want to borrow so that loan demand is given by the right-hand side of equation (31):

$$L^d = \frac{1+g}{1+r} D + B^g \tag{32}$$

We have omitted time subscript to indicate that we are evaluating the steady state. Meanwhile, the supply of loans can be derived in exactly the same way as before yielding:

$$L^s = -B^m = \frac{\beta}{1+\beta} \left(Y^m - D - T^m \right) - \frac{1}{1+\beta} \frac{Y^o - T^o}{1+r} \tag{33}$$

The only difference relative to our earlier expression for loan supply is that we now are keeping track of tax payments. The government's budget constraint is the final equation needed to determine asset market equilibrium:

$$T^m + B^g + \frac{1}{1+g} T^o + (1+g) T^y = G + (1+r) \frac{1}{1+g} B^g \tag{34}$$

where G is government spending (normalized in terms of the size of the middle generation), and again we omit time subscript to indicate we are evaluating the steady state. The equilibrium real interest rate is once again the interest rate that equalizes loan supply L^s and loan demand L^d and takes the same form as we saw previously in Section 2 and illustrated by Figure 1. The key difference is that now the real interest rate can be affected by fiscal policy, shifting the loan

[21] Another consideration against raising the inflation target and accommodating a negative natural rate of interest are concerns that very low rates could spur asset price bubbles and raise financial stability concerns. See Galí (2014) and Asriyan et al. (2016) for a discussion.

supply and loan demand curves in Figure 1. A fiscal policy regime corresponds to a choice of the level and distribution of taxation and government spending (T^o, T^m, T^y, G, B^g) subject to the government's budget constraint. The overall effect of fiscal policy on the real interest rate then depends on the contribution of all the fiscal variables. Equating (32) and (33), and taking account of the budget constraint (34), we have two equations and six unknown variables ($T^o, T^m, T^y, G, B^g, 1+r$). Hence, we need four restrictions on the fiscal policy instruments to determine the interest rate. Previously, we implicitly assumed that $T^o = T^m = T^y = 0$, $G = 0$ which implies $B^g = 0$ from the government budget constraint, leaving equations (32)-(33) to pin down the real interest rate $1 + r$.

Consider now a more general policy regime that will help us clarify a number of results.[22] The tax on the young and government spending are exogenously given by $T^y = T^*$ and $G = G^*$. Similarly, the overall level of *real* government debt is exogenously given by B^g_*. Finally, we assume that taxes on the middle aged and the old satisfy the following condition:

$$T^m = \frac{1}{\beta}\frac{1}{1+r}T^o = T \tag{35}$$

where the level of taxation T adjusts so that the budget constraint (34) is satisfied. The distribution of taxation condition given by (35) ensures that now there is no effect of fiscal policy on the supply of loans given by (33).

We first consider an increase in public debt B^g_*. Under the fiscal rule we have considered, an increase in public debt only affects loan demand, shifting out the demand for debt and raising the natural rate. In this respect, increasing government debt is a natural way of avoiding a secular stagnation.[23] Who receives the proceeds from this increase in government debt? The conditions above show that this does not matter so long as equation (35) is satisfied. The increase in government debt could be directed to the young, towards government spending, or distributed to the middle aged and the old in accordance with the fiscal rule (35).

We can similarly explore the effect of increasing government spending, funded via taxes, or various tax redistribution schemes. The effect of those policies can be gauged by analyzing how the policy shifts loan demand and loan supply in Figure 1. We will be a bit more specific about these type of policy experiments in the full lifecycle model.

A critical reason for why the increase in government debt raised the natural rate of interest was that it was expected to be permanent. To make this clear, consider the following policy regime.

[22] While this particular policy regime is clearly special, we think it helps illustrate relatively clearly how fiscal policy operates in this environment.

[23] The ability of an increase in the public debt in an OLG economy to undo the effect of credit frictions (that is, the effect of borrowing constraints) is similar to examples presented in Woodford (1990).

Table 1: Government purchases multiplier at zero lower bound

Financing	Multiplier	Value
Increase in public debt	$\frac{1+\beta}{\beta}\frac{1}{1-\kappa\psi}$	>2
Tax on young generation	0	0
Tax on middle generation	$\frac{1}{1-\kappa\psi}$	>1
Tax on old generation	$-\frac{1+g}{\beta}\frac{1}{1-\kappa\psi}$	<0

$G_t = T_t^y = B_{t-1}^g = 0$ and:

$$T_t^m = -B_t^g$$
$$T_{t+1}^o = (1+r_t)\, B_t^g$$

The thought experiment here is that an increase in the public debt results in a lump sum transfer to the middle aged in period t. The old are then taxed by the same amount in the next period (plus interest) to bring the debt down to its original level.[24] Therefore, this increase in public debt is only temporary. It is now easy to show that debt is completely irrelevant: the increase in debt in period t is met by a increase of the middle aged of their own savings to pay off the future tax. The point is that the effect of an increase in the public debt considered in the previous policy regime depends critically on agent's expectations about future fiscal policy. In particular, it depends upon the expectation of the middle aged that they will not be taxed to pay down the debt in the future.[25]

So far, we have only considered the effect of fiscal policy on the real interest rate in the endowment economy. Our results, however, carry over to the full model with endogenous production. Fiscal policy that leads to a change in the real interest rate in the endowment economy is equivalent to a policy that changes the natural rate of interest in the model with production. It therefore corresponds directly to a shock that shifts in the AD curve, as shown in right-hand panel of Figure 5 displaying the effect of an increase in government spending via debt issuance. Thus, fiscal policies that raise the the natural rate of interest correspond to an outward shift in the aggregate demand curve, while policies that reduce the natural rate correspond to an inward shift of the AD curve. An important new result also emerges when we consider the way in which fiscal policy shifts aggregate demand as shown in right-hand panel of Figure 5. In the case of monetary policy, an increase in the inflation target only allowed for the possibility of a "good equilibria" without explicitly ruling out the secular stagnation equilibria. In the case of fiscal policy, however, as we see in Figure 5, the secular stagnation equilibria can be eliminated since fiscal policy shifts the entire AD curve.

[24]For simplicity, we here abstract from population growth.

[25]Since money and public debt are perfect substitutes at the zero lower bound, a *temporary* helicopter drop would also have the same effect. The natural rate is unaffected.

To derive some analytic results for the effect of fiscal policy on output, let us generalize equation (26) by combining equations (32) and (33), together with equations (14), (22) and assuming $i = 0$, to yield:

$$Y = D + T^m + \frac{1+\beta}{\beta} B^g + \left(\frac{(1+g)(1+\beta)}{\beta} D - \frac{1}{\beta} T^o \right) \Pi \tag{36}$$

where we now see how fiscal instruments directly affect aggregate demand at a zero interest rate. Tracing out exactly how aggregate demand shifts requires being specific about the policy regime. In Table 1, we derive analytically the steady state multiplier of government spending at the zero lower bound under different financing conditions, under the policy regime we specified before (except we relax equation (35) when government spending is financed by the middle aged or old).[26] Observe that, away from the ZLB, the multiplier is zero since labor is supplied inelastically; once all workers are employed, government purchases will reduce private consumption one to one without any effect on output. At the zero bound, however, the multiplier is generally different from 0 as shown in Table 1.[27]

First, we consider the case in which spending is financed via an issuance of public debt. Financing via an increase in the public debt results in the largest multiplier shown in Table 1. Using the formula in Table 1, we see that because β is less than one, and κ can be no lower than zero (when wages are perfectly fixed), this multiplier has to be larger than 2. As we increase the value of κ, the multiplier becomes larger and can even be unboundedly large - a similar result as found in Christiano, Eichenbaum and Rebelo (2011) in the context of the standard New Keynesian model.

However, in contrast to the NK model, government spending multipliers are not always positive at the ZLB. The key issues is whether fiscal expansions reduce the saving glut in a secular stagnation which will depend on how government spending is financed. If instead of being financed by increasing debt, spending is financed via a tax on young households, the multiplier is zero. The collateral constraint is binding in equilibrium so that the young will cut their consumption by exactly the same amount as they are taxed. An increase in government purchases then simply substitutes for existing consumption by the young, leaving output unchanged. If an increase in spending is financed via a tax on middle-generation households, the purchases multiplier is still positive but smaller than if financed via debt as shown in Table 1. We see from the analytic expression that this multiplier always has to be greater than one. Finally, if government

[26] The parameters κ and ψ - the slope of the AS and AD curves respectively - are given by:

$$\kappa = \frac{1-\alpha}{\alpha} \frac{1-\gamma}{\gamma}$$
$$\psi = \frac{1+\beta}{\beta} (1+g) D$$

[27] In computing the multiplier, we consider only a small increase in spending (so the zero bound is still binding) and a linear approximation of the model around a zero inflation steady state to compute this statistic.

purchases are financed via a tax on the old, the multiplier is negative. This negative multiplier obtains because the old will cut their spending one for one with the tax (thus offsetting the higher spending by the government). Meanwhile the middle age will now increase their saving due to higher future taxes which reduces aggregate demand.

Nevertheless, the model suggests a relatively positive picture of fiscal policy as it can lead to an increase in demand via either debt policy or tax and spending actions.

7 A Quantitative Lifecycle Model

This section generalizes our three-period model to a full quantitative lifecycle model that incorporates a number of features that are important for a quantitative evaluation. Our chief result is that this lifecycle model can generate a permanently negative *natural rate of interest* under a standard parameterization that captures salient features of the US data. In addition, we use our model to explain the decline in interest rates since 1970 and decompose the factors responsible for this decline. Moreover, we consider the quantitative magnitude of changes in exogenous processes like productivity growth or in fiscal policy to return the natural rate of interest back to positive levels.

We outline the new elements and assumptions in the full lifecycle model, leaving a complete description to the Appendix. As the nominal frictions of the economy are the same as in the previous section, we focus only on the real side of the economy. Our model is closely related to Auerbach and Kotlikoff (1987) and Ríos-Rull (1996). In addition to longer life spans we add, among other things, a realistic age-income profile, uncertainty about death, a bequest motive, a more general CES production function with capital, and monopolistically competitive final good firms.

The economy consists of a large number of households with identical utility. Households enter economic maturity at age 26 after which they work, consume, have children, and participate in asset markets. Households pass away at age J, which we take to be 81 years. Individuals have children at age 26, and the population growth rate is determined by the total fertility rate (TFR) of every family. Households face a probability of dying stochastically before reaching maximum age J. The probability of surviving between age j and $j+1$ is denoted by s_j. The unconditional probability of reaching age j is denoted with a superscript s^j.[28]

Individuals receive utility from two sources: (1) consumption, which is given by a time-separable constant elasticity of substitution (CES) utility function $u(\cdot)$ with an elasticity of intertemporal substitution parameter ρ, and (2) bequests, which are divided equally among all descendants. The bequest motive is also characterized by a CES function $v(\cdot)$ whose argument is assets held at the last period of life. The utility of bequests is multiplied by a parameter $\mu \geq 0$ which determines the strength of the bequest motive. Denoting consumption of household of age

[28] This can be calculated by the multiplicative series $s^j = \Pi_{m=0}^{j} s_m$.

j at time t by $c_{j,t}$ and the discount rate by β, a household that enters economic maturity at time t has lifetime expected utility:

$$U_t = \sum_{j=26}^{J} s^j \beta^j u\left(c_{j,t+j}\right) + s^J \beta^J \mu v\left(x_{J,t+J}\right)$$

The household of age j can trade in a real asset $a_{j,t}$ at time t which is used as productive capital. At time $t+1$, capital will pay a return r^k_{t+1} which is the rental rate of capital, and has a resell value (net of depreciation) $(1-\delta)\xi_{t+1}$, where ξ_{t+1} is the exogenous relative price of capital in terms of the consumption good. Each household has an identical exogenous labor productivity process, or human capital profile, denoted by hc_j, which changes with age. Household receive no wage income after retirement, which in our model occurs after age 65. We assume an inelastic labor supply, hence wage income is equal to the wage multiplied by the individual age specific labor productivity hc^j net of labor taxes $(1-\tau^w)$.

Households also receive income from the pure profits of firms, denoted by $\pi^f_{j,t}$ and we assume that profits are distributed proportionally to labor income. Finally, the household may receive a bequest $q_{j,t}$. Bequest received are zero at all times except at age fifty seven, while bequests made are zero at all time except at age J. Following Ríos-Rull (1996), we suppose that agents insure themselves against the idiosyncratic risk of early death via one-period annuity contracts.

The flow budget constraint of a household of age j at time t is:

$$c_{j,t} + \xi_t a_{j+1,t+1} + TFR \cdot x_{j,t} = (1-\tau^w) w_t hc_j + \pi^f_{j,t} + \left(r^k_t + \xi_t\left(1-\delta\right)\right)\left(a_{j,t} + q_{j,t} + \frac{1-s_j}{s_j} a_{j,t}\right)$$

Household can borrow against future income, and we impose a borrowing constraint of the same form as in our earlier model.[29]

$$a_{j,t} \geq \frac{D_t}{1+r_t}$$

There are two types of firms: final and intermediate goods producers. The final good producers produce a differentiated good Y^f_t. The final good composite is the CES aggregate:

$$Y_t = \left(\int_0^1 \left(Y^f_t\right)^{\frac{\theta_t - 1}{\theta_t}} df\right)^{\frac{\theta_t}{\theta_t - 1}}$$

Each final good producer uses Y^m_t of intermediate good to produce output via a linear technology: $Y^f_t = Y^m_t$. The presence of monopolistically competitive final goods firms allows for time-varying markup given by $\frac{\theta_t}{\theta_t - 1}$ and returns pure profits to the household due to monopoly rents. As in the data, we include pure profits in the capital share of income.

There is a perfectly competitive intermediate good sector that sells their production to the final goods sector. These firms hire workers at rate w_t and rent capital at rate r^k_t. They operate a

[29]The borrowing constraint d_t grows at the rate of productivity growth and the individuals earning potential over his life cycle.

Table 2: Parameters taken from the data and related literature

Panel A: Data	*Symbol*	*Value*	*Source*
Mortality profile	$s_{j,t}$		US mortality tables, CDC
Income profile	hc_j		Gourinchas and Parker (2002)
Total fertility rate	n	1.88	US Census Bureau
Productivity growth	g	0.65%	Fernald (2012)
Government spending	G	21.3%	CEA
Public debt	B_g	118%	Flow of Funds
Panel B: Related literature			
Elasticity of intertemporal substitution	ρ	0.75	Rios-Rull (1996)
Capital/labor elasticity of substitution	σ	0.6	Antras (2004)
Deprecation rate	δ	12%	Jorgenson (1996)

CES production function in labor and capital with an elasticity of substitution σ. The production function is:

$$Y_t^m = \left(\alpha K_t^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\left(A_t L_t\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

where A_t is exogenous, labor augmenting technological progress and $0 < \alpha < 1$.

The government spends an exogenous G_t and may accumulate debt. The budget constraint is

$$b_{g,t} = G_t + (1+r_t)b_{g,t-1} - T_t$$

where taxes are collected on labor income. We economize on notation by omitting real and nominal bonds as assets above; these assets enter in the same way as in the simpler model so that there is both a well defined real interest rate r_t on a risk-free one period real bond and a nominal interest rate i_t on nominal bonds.

Monetary policy and the wage rigidity is modeled as in previous sections. The full nonlinear model is solved numerically; we solve for both the stationary equilibrium and perfect foresight transition paths. We outline the computational details and the numerical solution algorithm in the Appendix.[30]

7.1 Calibration

We first calibrate our model to match the US economy in 2015. There remains considerable uncertainty about about the size of the output gap in 2015. We consider two polar cases: a zero output

[30]The equilibrium conditions consist of about 170 equations in stationary equilibrium. Transition dynamics are more complicated as the birthrate over the past 25 years becomes a state variable in that case and agents form expectations over the entire transition path.

gap in 2015 based on Stock and Watson (2012) and an alternative calibration where the output gap is -15% which is what Hall (2016) estimates as the deviation of output was from its pre-recession trend in 2015. We consider this latter case as a sensible lower bound on the possible size of the output gap.[31]. Consider first the case in which there is not output gap. Since the real interest rate in the US was -1.47% in 2015 this then implies that the natural rate of interest was at -1.47% as well. The model can then be parameterized without any reference to nominal frictions. This is our main benchmark, and we ask whether it is reasonable to expect this interest rate to correspond to the long-run steady state natural rate of interest. If the natural rate of interest is this low in steady state, then ZLB should be expected to be a recurrent problem. Consider now the alternative extreme of 15 percent output gap. We ask if our model can generate this outcome and the associated deviation of inflation from target in 2015.

Our parameters come from three main sources. The first is statistical data about US demographics and the economy that we can match directly, such as mortality data and fertility, productivity and the size of government debt. The second is calibrated parameters, that we take directly from related literature that has estimated them. The third set of parameters are chosen to match key moments in the data, such as the investment to output ratio. We will discuss the different sources in turn.

Tables 2 and 3 shows our list of benchmark parameter values. The first category of parameters are chosen from directly observed data. We choose mortality data from the Centers for Disease Control (CDC) to directly match US survival tables. The total fertility rate is taken from US Census Bureau data, and the retirement length is chosen to match the average years of retirement. Government debt to GDP and government spending to GDP are also chosen to match current values.[32] The rate of productivity growth is a key determinant of the real interest rate. Our baseline uses a productivity growth rate of 0.7% per year which we taken from Fernald et al. (2012). The wage profile hc_j is chosen to match the earnings profile estimated from the data by Gourinchas and Parker (2002).

In the second category, we choose another set of parameter based on related literature. We set the intertemporal elasticity of substitution, $\rho = 0.75$. This parameter has been estimated widely in the literature, with ranges between $.25$ and 1.[33] The depreciation rate comes from Jorgenson (1996) and of Economic Analysis (2004), who have extensive estimates of the depreciation rate of private and governmental nonresidential equipment.[34] The value of elasticity of substitution in

[31]To be clear, Hall (2016) does not interpret this gap as reflecting output gap of the kind we see in the model, but we still think that computing deviation form trend provides a convenient benchmark to contrast with no output gap

[32]We set government debt as the sum of federal and state and local debt as reported by the Council of Economic Advisors and the Census Bureau.

[33]For example, Cooley and Prescott (1995) sets $\rho = 1$, while Auerbach and Kotlikoff (1987) and Ríos-Rull (1996) set $\rho = .25$.

[34]Since our model does not explicitly incorporate housing, and thus abstracts from capital risk, our preferred spec-

Table 3: Parameters chosen to match targets

Targets	*Model/Data*	*Source*
Natural rate of interest	-1.47%	PCE FFR
Investment to output ratio	15.9%	NIPA
Consumer debt to output ratio	6.3%	Flow of Funds
Labor share	66.0%	Elsby (2013)
Bequests to output	3.0%	Lutz (2001)
Parameters chosen to match targets	*Symbol*	*Value*
Rate of time preference	β	0.98
Borrowing limit (% of annual income)	D	23.4%
Bequests parameter	μ	21.6
Retailer elasticity of substitution	θ	4.9
Capital share parameter	α	0.24

production, σ, has also been estimated widely in the literature, and generally falls between .4 and 1.[35] As these parameters are not directly observed in the data, we report in the Appendix 7 how our result change for different parameter choices, and will also comment on them below.

We choose the remaining parameters to match five key moments of the data as of 2015: the real interest rate -1.47 percent, the investment to output ratio of 15.9%, a consumer debt ratio 6.3% percent, a labor share of 66.0% percent, and a bequest to output ratio at 3 percent with sources listed in Table 3 and the associated footnotes.[36] The parameters chosen this way are the rate of time preference β, the debt limit D, the bequest parameter μ, the capital share parameter α, and retailer elasticity of substitution θ.

Generically, there is no one-to-one mapping between the remaining parameter and the targets. Hence, we jointly choose all parameters to match the model output to the targets. Nevertheless, the parameters above correspond relatively closely with the key moments we are trying to match. The rate of time preference β has a direct effect on the real interest rate; as β increases, the real interest rate falls. The debt limit most directly affects the level of consumer debt to output. The bequest parameter μ mostly directly affects the bequest to output ratio. Finally, capital share

ification uses depreciation applicable to equipment investment which is higher than for housing. We consider a lower value in the sensitivity analysis that follows.

[35]For example, Antras (2004) estimates it is between .4 and .9, Oberfield and Raval (2014) estimate it is .7, and Klum (2007) gives a value between 0.5 and 0.6

[36]We choose to target the investment to output ratio, rather than capital output ratio often used, as this ratio does not require adjustment of the past capital stock for changes in the relative price of capital goods - the investment to output is just a ratio of two nominal, readily measured quantities.

parameter α determines the investment to output ratio, while inverse profit share θ controls the labor share. Our calibration targets in Table 2 are hit to the first decimal.

7.2 Negative Real Interest Rates

Table 3 illustrates the first major quantitative result of our model. By choosing each of the parameters as shown in the table, the model is able to replicate the key targets from the data while generating a negative steady state real interest rate of -1.47 percent. Importantly, we can generate a substantial negative rate with parameters that are either directly taken from the data and/or fall into perfectly standard ranges observed in the literature, as we report in Appendix J. This suggest that relatively standard OLG model with endogenous capital can generate permanently negative real interest under standard parameter values.[37]

The fact that a standard calibration under full employment delivers a substantially negative natural rate of interest suggest that there is no reason a priori to expect a normalization of interest rates in the US. Pre-recession expectation for the long-run neutral real interest rate remained between 2% and 3%. As long-term rates have continued to fall, the Federal Reserve has adjusted downward its estimates of the neutral rate to closer to 1%.

Our calibration, however, suggests that given current productivity and demographic trends, these estimates for the long-run neutral rate still remain too optimistic, indeed our benchmark calibration has the steady state real interest rate at -1.47% which is 2.27% higher than the Federal Reserves estimate as of 2016.

Our results are robust to a variety of additional specifications for the three parameters we picked from the outside literature. Appendix J calculates the same moments in Table 3 for three alternate specifications: (1) setting depreciation to 8%, (2) setting the intertemporal elasticity of substitution $\rho = 1$, (3) setting the production elasticity parameter to 1 - the standard Cobb-Douglas production function. For each of these specifications, we again choose the parameters β, α, μ D, and θ to match the targets. The results for these alternate specifications are similar to those of our main specification. The model still hits the targets but with different values for the calibration parameters that still remain well within ranges commonly assumed in the literature. We report results for other experiments in the remainder of this section under these alternative parameterizations in Appendix J.

[37] In the Appendix J, we show the optimal consumption path for individuals over their lifecycle. Consumption tracks income over the early part of the lifecycle due to borrowing constraints, then declines gradually in order to save for retirement. This leads to the classic ”hump-shaped” consumption profile. For comparison purposes, this figure also includes the estimated consumption profile from the Consumer Expenditure Survey, as estimated by Gourinchas and Parker (2002). We also display the model's population pyramid in Appendix J as compared to current US pyramid. The fact that we are considering the stationary equilibrium is not innocuous. It does not take into account the dynamics of the aging of the baby boom population and its effect on interest rates as the boom filters through the population pyramid. To study these effects, we need to consider transition dynamics, which we turn to shortly.

While the model can clearly replicate permanently negative short-term real interest rates, it is also of interest to explore if it can explain the reduction observed in real interest rate over time and how a forecast of future real interest rate is affected by various of the ingredients of our calibration. Before getting there, however it is worth asking if the model can also replicate a scenario in which the natural rate of interest is negative enough that the zero bound is binding and the economy is in secular stagnation with output below potential. This is what we turn to next.

7.3 Secular Stagnation in a Quantitative Lifecycle Model

Although our initial calibration chooses parameters to match a zero output gap, there is evidence that output in the US remained significantly below potential in the last few years (see for example, Ball (2014), Reifschneider (2016)). Consistent with this point of view, inflation has been remained stubbornly lower than the target rate of 2%. Likewise, the Great Recession has seen a large decline in the employment to population ratio. We now ask whether our model can generate this outcome, and perhaps more importantly, can it generate a permanent recession of the form we showed in our simple model?

For this experiment, we keep the bequest parameter μ, capital share α, debt limit D, and retailer EIS θ unchanged from our benchmark calibration. There are two other remaining free parameter: First, the rate of time preference β, and second, the wage rigidity parameter γ. Observe that we did not need to take a stance on γ in our benchmark parameterization, as we assumed that there was no output gap, hence the wage norm was not binding. We jointly choose β and γ to match two moments of the US economy in 2015. We model the economy as in a secular stagnation, with inflation rate of 1.62% as in the data and output gap of -15%, corresponding to the deviation of output from its pre-crisis trend in 2015 as documented in Hall (2016) We view this as a reasonable lower bound on the output gap and a natural counterpart to our zero output gap experiment. A γ of .91 and β 0.99 match the output gap of -15% and the US PCE inflation rate of 1.62%. Our calibrated value of γ falls within the range of degree of wage rigidity estimated by Schmitt-Grohé and Uribe (2015). The value β is also consistent with common estimates from the OLG literature.

We can construct aggregate demand and supply curves to represent the secular stagnation stationary equilibrium as depicted in Figure 6. The aggregate demand and aggregate supply lines cross at an output gap of -15%. The natural, flexible price interest rate under this calibration is -2.2%. The inflation target of 2% is too low to allow the nominal interest rate to remain above zero in an economy with no output gap. As a result, the economy is drawn into the secular stagnation equilibrium. The equilibrium real interest is -1.62% and the inflation rate is 1.62% - 40 basis points below target. Investment to output, bequests to output, labor share, and consumer debt to output continue to closely match the US moments.[38]

[38] The investment to output ratio, consumer to debt ratio, labor share and bequest to output remain very close to

Figure 6: Secular stagnation equilibrium

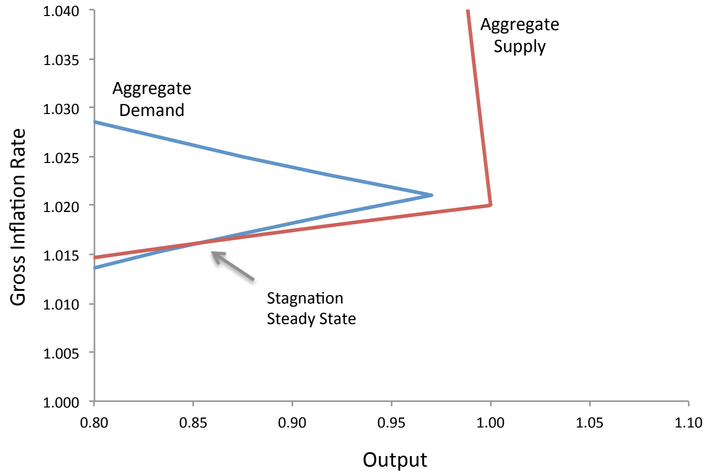


Figure 6 represents the second major quantitative finding of the paper. It suggest that the model can replicate a permanent stagnation with relatively standard parameterization, of precisely the same form as we have shown in the illustrative model. As in the 3-period model, a surplus of savings over investment opportunities pushes the equilibrium interest rate negative. If the inflation target is insufficient to prevent the zero lower bound from binding and nominal wage rigidities bind, a secular stagnation emerges.

7.4 Decline in Interest Rates since 1970

We can use our model to quantitatively explore the fall in the real interest rate observed since 1970 and assess the relative contribution of different factors. As discussed in the introduction, 1970 is a natural starting point as key economic trends relevant for the interest rate began shifting after this year. In particular, over this time period, the US experienced a significant increase in life expectancy, a fall in birthrate, a fall in TFP growth, an acceleration in the fall of the relative price of investment goods and a reduction in labor share. Also, over this period, there was a significant rise in government debt.

We begin by comparing stationary equilibria, before turning to transition dynamics. Relative to our previous calibration, we adjust the a subset of the first set of parameters taken directly from the data (see Table 2) to their relevant counterparts from the 1970s. These are documented in Panel A in Table 3 which compares the change in these parameters over this time period.[39] Panel B of Table 3 documents another important phenomena, which is the fall in relative price of investment

their value in Table 3, hence we choose not to reparameterize the model for μ, α, D, and θ relative to the benchmark calibration.

[39]For simplicity we list in the table the increase in life expectancy, but in the model we use the change in age dependent probability of death which has more information.

Table 4: Change in parameters from 1970 to 2015

Panel A: Data	*1970*	*2015*
Mortality rate	70.7	78.7
Total fertility rate	2.8	1.9
Productivity growth	2.02%	0.65%
Government debt (% of GDP)	42%	118%
Panel B: Relative Price of Investment		
Relative price of investment (index 100=2015)	130	100
Panel C: Change in targets		
Consumer debt to output ratio	4.2%	6.3%
Labor share	72.4%	66.0%

over this period which is also incorporated in our 1970's simulation. Finally, panel C document changes in two of the targets we used in our previous parameterization, namely the higher labor share in 1970s as well as the lower consumer debt.

We initially chose β, α, μ, D, and θ to match the moments of the 2015 data. For the 1970 experiment, we keep the utility and production parameters (β, α, μ) unchanged. We adjust the collateral constraint D, to match the lower consumer debt to output ratio observed in 1970 and the profit share θ to match the higher labor share observed in 1970's. The 1970 values for these two parameters are D=0.144, and θ=7.93.[40]

Table 5 shows the results simulating the model with under the 1970's calibration, and is the third major quantitative finding of the paper. By our choice of D and θ the model matches the consumer debt to output ratio exactly as well as the labor share. Of greatest interest is to see if the 1970 calibration can match the moments that we did not explicitly target, namely the real interest rate and investment to output ratio. As we can see, by feeding in the 1970's values for the forcing variables in the model, it can indeed account for the positive real interest rate observed in 1970, predicting a real interest rate in the stationary equilibrium of 2.55%, slightly below the 2.62% observed in the data. Similarly the model predicts investment output ratio slightly higher than what is observed in the data. Overall, the model does a reasonable job of explaining the fall in the real interest rate observed over the past 45 years by using observed changes in productivity, demographics, relative price of investment, and credit constraints.

Table 6 decomposes the contribution of each of these factors over the past 45 years. We change each parameter from it's steady state value in 2015 to its steady state value in 1970 holding all other

[40] We keep the three parameters we calibrated (σ, ρ, and δ) unchanged as well as the income profile.

Table 5: Simulation results for 1970

Moment	*Model*	*US data*
Natural rate of interest	2.55%	2.62%
Investment to output ratio	19.0%	16.8%
Consumer debt to output ratio	4.2%	4.2%
Labor share	72.4%	72.4%

Table 6: Decomposition of decline in natural rate of interest: 1970-2015

Forcing variable	Δ in r	% of total Δ
Total interest rate change	-4.02%	100%
Mortality rate	-1.82	43%
Total fertility rate	-1.84	43%
Productivity growth	-1.90	44%
Government debt (% of GDP)	$+2.11$	-49%
Labor share	$-.52$	12%
Relative price of investment goods	-0.44	10%
Change in debt limit	$+.13$	-3%

parameter constant and examine the effect of this change on the real interest rate. For example, changing productivity growth from its 2015 level of 0.65% per year to it's 1970 level of 2.02% results in an increase in the steady state real interest rate of 1.9 percentage points. The table shows a decomposition of the relative importance of all the other factors.[41] The reduction in fertility, mortality, and the rate of productivity growth over this time period play the largest role in the decrease in real interest rates. The main factor which has tended to counterbalance these forces is an increase in government debt. Changes in the labor share and the relative price of investment goods play a smaller role in explaining the decline in real interest rates. Similarly the increase in consumer debt limit does not have a significant effect on the evolution of the interest rate in this parameterization.[42]

[41] The decomposition is calculated by first adding up the total change in the real interest rate from all of these factors, and then dividing the individual change from changing a particular forcing variable by the entire change. Numbers may not sum to unity due to interaction effects.

[42] As our calibration only takes account of consumer debt, we suspect that this force may play a substantially larger role once one takes account of housing purchases and the associated debt, along with firms borrowing and lending. Together these forces would have tended to increase the real interest rate in the past 45 year, although lowered them during the crisis. Working reducing real interest rates over the past 45 is the increase in inequality observed during this period, which we have not taken account of.

Table 7: Raising the natural rate of interest to 1%

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	3.28
Government debt (% of GDP)	118%	215%
Productivity growth	0.65%	2.43%
Relative price of investment goods	1.00	2.43

7.5 Raising the Natural Rate of Interest

Our initial steady state analysis calibrates the steady state natural interest rate to be -1.47% in 2015. If accurate, a natural rate this negative poses a challenge for policy makers; small decreases in the natural rate will cause the zero lower bound to bind meaning that downturns may be sharper and more persistent. Table 7 considers an alternative thought experiment taking as given that the natural rate is -1.47% in 2015. It asks, what economic conditions would be needed to increase the steady state real interest rate to a positive territory of 1 percent? While this target is somewhat arbitrary, we find it to be useful benchmark. With the Federal Reserve's inflation target of 2 percent, a natural rate of 1% would give policymakers a reasonable room to respond to negative shocks that otherwise lead to a binding ZLB. It also corresponds with the current expectations of the FOMC about long-run real interest rate. An alternative way of formulating our experiment is to ask which forcing variables in 2015 would one need to change to be consistent with the current FOMC projections.

As Table 7 suggests, substantial changes in the underlying fundamentals are needed in order to increase the natural rate to 1%. Given current demographic trends, it is implausible that fertility would reverse its decline and increase so drastically from 1.88 to 3.28. An increase in immigration could make up a portion of the difference. It would also be challenging to increase productivity growth to 2.43% per year given the headwinds to productivity noted by Gordon (2016) and given that productivity growth has rarely exceeded 2% since 1970.

Of particular interest is the extent to which an increase in government debt could sustain a higher natural rate of interest given that this is a obvious policy level. As Table 7 shows, government debt would need to double to roughly 215% of GDP. Such high number raises questions about the feasibility of this policy, for we have not modeled any costs (e.g. a probability of increasing real interest rate in the future for reason outside the model which imposes taxation costs) or limits on the government's ability to issue risk-free debt - an assumption that may be strained at such high level of government debt. While these results suggest that there are several reforms that would tend to increase the natural rate of interest, the menu of options does not paint a particularly rosy picture relative to the alternative of raising the inflation target of the central bank. In the

Appendix we document how these result change for different choices for three calibrated values form the outside literature. While the exact numbers do change the overall tenor of the result is the same.

7.6 Transition Dynamics

So far, we have confined our analysis to stationary equilibria. While this is one natural benchmark, we can also consider transition dynamics. This requires taking a strong stance on agents expectations during this transition. Here we document numerical experiments in which it is assumed that the economy was at a stationary equilibria in 1970 and then project the model forwards, assuming agents have perfect foresight about the path of exogenous processes. Overall, we find that the results are largely consistent with the steady state comparisons already documented.

We feed into the model the dynamic paths for each of the forcing variables in Table 6, and calculate the resulting equilibrium. We document the time series for each forcing variables in the Appendix. After 1970, there is perfect foresight about how each of these variables will change, as well as perfect foresight over all endogenous variables. For example, in 1970 all agents presently alive will realize that there will be a productivity slowdown over the next 40 years, and will adjust their optimal decisions accordingly. The forcing variables are set to their final steady state values after the year 2015.

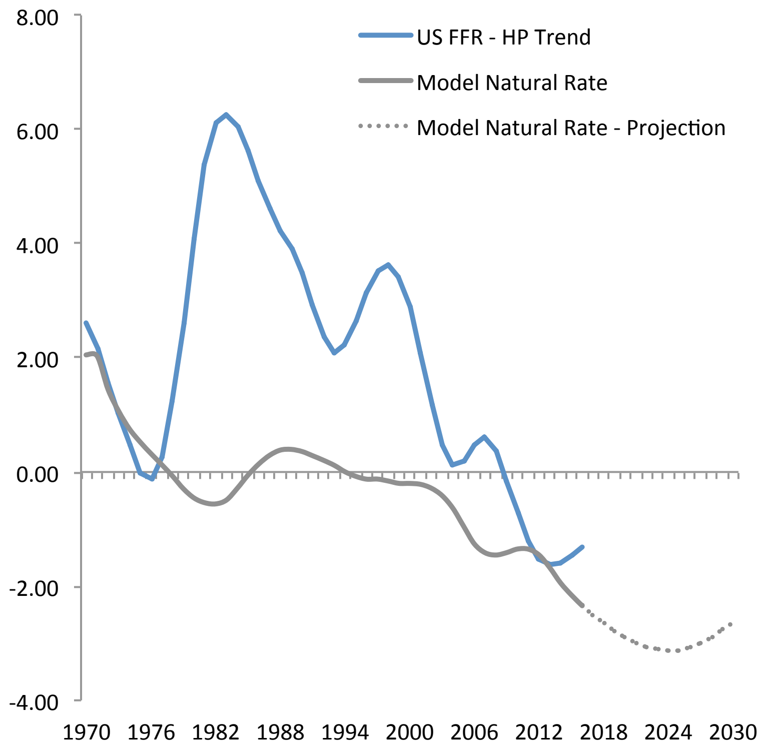
Since people enter the labor force at the age of 26, the birthrates from 1945 to 1970s are key state variables when people make forecast about the future path of the economy in 1970s. For our projection for 1970 onwards, we thus use as input birthrates in the US economy from 1945 onwards, to measure the size of each incoming generation. The birthrate date from 1945 are reflected below the solid line in the age pyramid for the US economy in figure 8, but in 1970 those generation are economically inactive, only entering the labor force gradually, an issue we return to shortly.

Figure 7 shows the full transition path for the real interest rate. The 1970 real interest rate is 2.55% and declines non-monotonically throughout the time period until it reaches -1.0% in 2015. The interest rate exhibits a brief recovery in the mid to late 1990s as productivity growth increases, but this subsides by the mid 2000s. Note that the model projects that the interest rate will continue to decrease until it hits a nadir in 2020. After 2020, there are cycles in the real interest rate due to the echo effects of the baby boom. The economy gradually converges to the final steady state interest rate of -1.47%.[43]

One interesting observation is that our model predicts a more rapid decline in the real interest rate that observed in the data. A possible explanation for this, which we leave for future research,

[43]One factor we do not focus on in these transition paths is debt deleveraging. While the evidence and theory certainly suggests there was a significant deleveraging shock in 2008, our model as of yet does not include a housing sector. Since housing debt is a significant portion of consumer debt, we leave quantitative work of the effect of debt deleveraging on interest rates to future work.

Figure 7: Transition path of the natural rate of interest



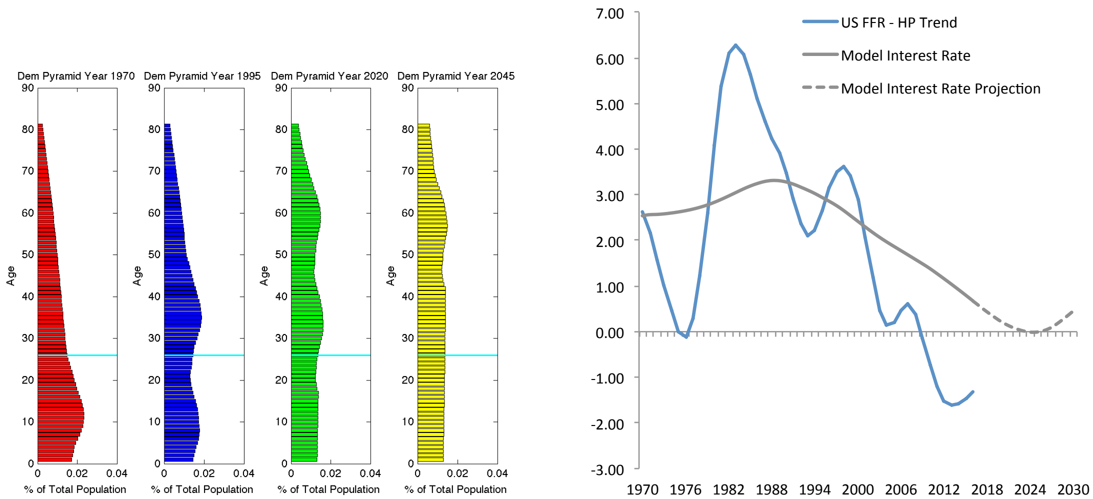
is suggested by Summers (2014). He hypothesizes that the decline in the interest rate since the late 1990's was masked by both the tech and housing bubble, so that the true natural interest rate has only been observed post crisis. This could be modeled as an exogenous increase and fall in the collateral constraint D in our model, but we leave this for future research.

Our model does not generate an investment boom even though real interest rates are declining. In fact, investment actually falls in our model, from 19.0% in 1970 to 15.9% in 2015. Although lower interest rates will tend to lead to a higher capital to output ratio and thus a higher investment rate, there are several counterbalancing forces in the model. In particular, a decrease in the rate of population growth and productivity growth mean lead directly to a lower investment to output ratio holding constant the capital to output ratio. Finally, our relatively inelastic production CES parameter means that the steady state K/Y ratio stays relatively constant for a given interest rate, which also dampens the effect on investment.

7.7 Effects of the Baby Boom

We now separate the effects of the baby boom on real interest rates. As the baby boom started in 1945, the children of the baby boom did not enter economic maturity until they reached age 26 in 1970; thus, we are able to look at the impact of these demographic changes on the evolution of

Figure 8: Baby boom and bust



interest rates starting from our model economy in 1970.

The left-hand panel of Figure 8 shows the demographic pyramid for the model during the transition years. In 1970 the economically active population is assumed to be in steady state, thus the pyramid above the solid line is a triangle with the anticipation of the bulge below the line to enter the labor force. Twenty five years later 25 (1995), the bulge now shows up in the working age population. Fifty years later, 2020 the baby boom generation is retired. There is also a second bulge in the population pyramid due to the children of the baby boomers. By the year 100 the population pyramids are approaching steady state.

The right-hand panel of Figure 8 shows the interest rate transition for the baby boom exercise. From 1970 until 2000 the baby boom tends to increase real interest rates due to an increase in the rate of population growth. After the baby boom comes the baby bust, and fertility rates drop as the boomers enter retirement. This leads to a sharp drop in real interest rates. The echo effects result in cycles in the real interest rate until it reaches its final steady state value of around 1%.

8 Conclusion

In this paper, we formalize the secular stagnation hypothesis using both a simple three-period model and a 56 period quantitative lifecycle model. We define secular stagnation as a persistent decline in the natural rate of interest - the real interest rate consistent with full employment. In an overlapping generation model, the natural rate of interest may be negative raising the possibility of a chronically binding zero lower bound absent any increase in the inflation target. With a chronically binding ZLB, the economy may suffer from persistent output gaps or, even if full-employment is somehow achieved, the prospect of sharper and more severe recessions in response to negative shocks.

Our simple three period model highlights the basic forces that can push the natural rate of interest negative. Reductions in productivity growth, a fall in population growth, rising income inequality, falling prices of capital goods, and deleveraging can all reduce the natural rate of interest. Using our quantitative lifecycle model, we incorporate several of these forces and show, under a standard calibration, that, at current levels of productivity growth and taking into account aging due to the baby boom and bust and falling mortality rates, the natural rate of interest is significantly negative. Demographic factors, which are accurately forecasted and unlikely to change unexpectedly will provide little support for the natural rate in coming years, leaving an unexpected acceleration in productivity as the only factor that may sharply raise the natural rate in coming years.

Our model also illustrates the importance of active policy response - particularly a fiscal response - in a secular stagnation. In stark contrast to existing New Keynesian models, a recovery at some future date is not assured and effectiveness of forward guidance is drastically curtailed. Low or negative natural rates of interest suggest an obvious policy response of adopting a higher inflation target, but, as we showed, higher inflation targets do not foreclose the secular stagnation steady state. Fiscal policy, by directly managing the natural rate of interest, does not face this shortcoming.

On drawback of raising the inflation target, as Summers (2014) argues, is negative real interest rates may generate bubbles. In a similar vein, Bernanke (2015) argues against the secular stagnation hypothesis on theoretical grounds: the existence of very long-lived productive assets like farm land places a floor on real interest rates. These arguments generally rely on dynamic efficiency argument that are closely tied with overlapping generations models. Dynamic efficiency asks whether an economy overaccumulates capital. The typical dynamic efficiency condition is that $r > n$ where n is population growth; given positive population growth, a negative real interest rate would imply overaccumulation of capital. Relatedly, Gomme, Ravikumar and Rupert (2015) note, the observed average product of capital (using data from the national income accounts) remains positive and exhibits no pronounced trend over the postwar period.

Several considerations should be noted on these points. First, in our secular stagnation steady state shown in the simple model, the equilibrium real interest *exceed* the natural rate of interest. That is, the source of the demand shortfall is the interest rates are too high relative to what is needed to ensure full-employment; the standard dynamic efficiency condition ($r > n$) is satisfied. Second, even though we index the wage norm to the inflation target (thereby allowing for negative interest rates in the secular stagnation steady state in both the three-period and 56-period calibrations) this standard dynamic efficiency condition continues to hold in our quantitative model. In this model, with capital, dynamic efficiency requires that $MPK - \delta$ ¿ n. Because of the presence of monopoly power and markups, this condition is also satisfied in the 2015 calibration of our quantitative model. That is, our stationary equilibrium is dynamically efficient.

Finally, we can also measure the average product of capital in the same was as Gomme, Ravikumar and Rupert (2015). Here, we take payments to net domestic product less labor compensation divided by the nominal capital stock. This measure of average product show little change from the 1970 stationary equilibrium to 2015 stationary equilibrium. Here, the rise in markups needed to account for the fall in the labor share partially offsets the fall in the real interest rate to keep average product from falling markedly. Overall, while these issue deserve further investigation, we do not think they are, at this stage, strong evidence against the mechanisms we have emphasized here. We leave a fuller investigation of these issues to future work.

References

Abel, Andrew B., N. Gregory Mankiw, Lawrence H. Summers, and Richard J. Zeckhauser. 1989. "Assessing Dynamic Efficiency: Theory and Evidence." *Review of Economic Studies*, 56(1): 1–19.

Asriyan, Vladimir, Luca Fornaro, Alberto Martin, and Jaume Ventura. 2016. "Monetary Policy for a Bubbly World." National Bureau of Economic Research.

Auerbach, Alan J., and Laurence J. Kotlikoff. 1987. *Dynamic Fiscal Policy.* Cambridge University Press.

Ball, Laurence M. 2014. "Long-term Damage from the Great Recession in OECD countries." National Bureau of Economic Research 20185.

Barattieri, Alessandro, Susanto Basu, and Peter Gottschalk. 2010. "Some Evidence on the Importance of Sticky Wages." National Bureau of Economic Research.

Benhabib, Jess, Stephanie Schmitt-Grohé, and Martın Uribe. 2001. "The Perils of Taylor Rules." *Journal of Economic Theory*, 96(1): 40–69.

Benigno, Gianluca, and Luca Fornaro. 2015. "Stagnation Traps." Mimeo, London School of Economics.

Bernanke, Ben S. 2015. "Why Are Interest Rates so Low, Part 2: Secular Stagnation."

Bewley, Truman F. 1999. *Why Wages Don't Fall During a Recession.* Harvard University Press.

Bhattarai, Saroj, Gauti B. Eggertsson, and Raphael Schoenle. 2014. "Is Increased Price Flexibility Stabilizing? Redux." National Bureau of Economic Research 19886.

Brynjolfsson, Erik, and Andrew McAfee. 2014. *The Second Machine Age: Work, Progress, and Prosperity in a Time of Brilliant Technologies.* WW Norton & Company.

Bullard, James. 2010. "Seven Faces of the Peril." *Federal Reserve Bank of St. Louis Review*, 92(5): 339–52.

Christiano, Lawrence, Martin Eichenbaum, and Sergio Rebelo. 2011. "When Is the Government Spending Multiplier Large?" *Journal of Political Economy*, 119(1): 78–121.

Cooley, Thomas F, and Edward C Prescott. 1995. "Economic growth and business cycles." *Frontiers of business cycle research*, 1.

Del Negro, Marco, Marc P Giannoni, and Christina Patterson. 2012. "The Forward Guidance Puzzle." *Federal Reserve Bank of New York Staff Report*, , (574).

Delong, Brad. 2014. "What Market Failures Underlie our Fears of Secular Stagnation?" Blog post.

Dynan, Karen E., Jonathan Skinner, and Stephen P. Zeldes. 2004. "Do the Rich Save More?" *Journal of Political Economy*, 112(2): 397–444.

Eggertsson, Gauti B., and Michael Woodford. 2003. "The Zero Bound on Interest Rates and Optimal Monetary Policy." *Brookings Papers on Economic Activity*, , (1): 139–234.

Eggertsson, Gauti B, and Paul Krugman. 2012. "Debt, Deleveraging, and the Liquidity Trap: A Fisher-Minsky-Koo Approach." *The Quarterly Journal of Economics*, 127(3): 1469–1513.

Eggertsson, Gauti B., and Sanjay R. Singh. 2016. "Log-linear Approximation versus an Exact Solution at the ZLB in the New Keynesian Model." National Bureau of Economic Research 22784.

Fallick, Bruce, Michael Lettau, and William Wascher. 2011. "Downward Nominal Wage Rigidity in the United States During the Great Recession." Mimeo, Federal Reserve Board.

Fernald, John, et al. 2012. "A quarterly, utilization-adjusted series on total factor productivity." *Federal reserve bank of San Francisco working paper*, 19: 2012.

Galí, Jordi. 2014. "Monetary Policy and Rational Asset Price Bubbles." *American Economic Review*, 104(3): 721–752.

Garga, Vaishali, and Sanjay Raj Singh. 2016. "Output Hysteresis and Optimal Monetary Policy." Mimeo. Brown University.

Gomme, Paul, B Ravikumar, and Peter Rupert. 2015. "Secular Stagnation and Returns on Capital." *Federal Reserve Bank of St. Louis Economic Synopses*, , (19).

Gordon, Robert J. 2015. "Secular Stagnation: A Supply-side View." *American Economic Review*, 105(5): 54–59.

Gourinchas, Pierre-Olivier, and Jonathan A. Parker. 2002. "Consumption over the Life Cycle." *Econometrica*, 70(1): 47–89.

Guerrieri, Veronica, and Guido Lorenzoni. 2011. "Credit Crises, Precautionary Savings, and the Liquidity Trap." National Bureau of Economic Research.

Hall, Robert E. 2014. "Quantifying the Lasting Harm to the US Economy from the Financial Crisis." In *NBER Macroeconomics Annual 2014, Volume 29*. University of Chicago Press.

Hall, Robert E. 2016. "The Anatomy of Stagnation in a Modern Economy."

Hansen, Alvin. 1939. "Economic Progress and Declining Population Growth." *American Economic Review*, 29(1): 1–15.

Hausman, Joshua K., and Johannes F. Wieland. 2014. "Abenomics: Preliminary Analysis and Outlook." *Brookings Papers on Economic Activity*, , (1): 1–63.

Jorgenson, Dale W. 1996. "Empirical studies of depreciation." *Economic inquiry*, 34(1): 24–42.

Kocherlakota, Naryana. 2013. "Impact of a Land Price Fall when Labor Markets are Incomplete." Mimeo, Federal Reserve Bank of Minneapolis.

Krugman, Paul R. 1998. "It's Baaack: Japan's Slump and the Return of the Liquidity Trap." *Brookings Papers on Economic Activity*, 2: 137–205.

Krugman, Paul R. 2013. "Secular Stagnation, Coalmines, Bubbles, and Larry Summers." Blog post.

Krusell, Per, Lee E. Ohanian, José-Víctor Ríos-Rull, and Giovanni L Violante. 2000. "Capital-skill Complementarity and Inequality: A Macroeconomic Analysis." *Econometrica*, 68(5): 1029–1053.

McKay, Alisdair, Emi Nakamura, and Jón Steinsson. 2015. "The Power of Forward Guidance Revisited." National Bureau of Economic Research.

Mehrotra, Neil. 2012. "Fiscal Policy Stabilization: Purchases or Transfers?" Mimeo, Columbia University.

Mian, Atif, and Amir Sufi. 2012. "What Explains High Unemployment: The Aggregate Demand Channel." National Bureau of Economic Research.

of Economic Analysis, Bureau. 2004. "BEA Depreciation Estimates."

Reifschneider, David. 2016. "Gauging the Ability of the FOMC to Respond to Future Recessions." Mimeo. Federal Reserve Board.

Ríos-Rull, José-Víctor. 1996. "Life-cycle Economies and Aggregate Fluctuations." *Review of Economic Studies*, 63(3): 465–489.

Schmitt-Grohé, Stephanie, and Martin Uribe. 2011. "Pegs and Pain." National Bureau of Economic Research.

Schmitt-Grohé, Stephanie, and Martın Uribe. 2013. "The Making of Great Contraction with a Liquidity Trap and a Jobless Recovery." Mimeo, Columbia University.

Schmitt-Grohé, Stephanie, and Martın Uribe. 2015. "Downward Nominal Wage Rigidity, Currency Pegs, and Involuntary Unemployment." *Journal of Political Economy*, Forthcoming.

Shimer, Robert. 2012. "Wage Rigidities and Jobless Recoveries." *Journal of Monetary Economics*, 59(S): S65–S77.

Stock, James H, and Mark W Watson. 2012. "Disentangling the Channels of the 2007-09 Recession." *Brookings Papers on Economic Activity*, 120–157.

Summers, Lawrence. 2013. "Why Stagnation Might Prove to be the New Normal." The Financial Times.

Summers, Lawrence H. 2014. "US economic prospects: Secular stagnation, hysteresis, and the zero lower bound." *Business Economics*, 49(2): 65–73.

Taylor, John B. 2014. "The Economic Hokum of 'Secular Stagnation'." The Wall Street Journal.

Thwaites, Gregory. 2015. "Why are Real Interest Rates so Low? Secular Stagnation and the Relative Price of Investment Goods." Mimeo. London School of Economics.

Tobin, James. 1972. "Inflation and Unemployment." *American Economic Review*, 62(1/2): 1–18.

Werning, Ivan. 2012. "Managing a Liquidity Trap: Monetary and Fiscal Policy." Mimeo, Massachusetts Institute of Technology.

Williams, John C. 2016. "Monetary Policy in a Low R-star World." *FRBSF Economic Letter*, 23.

Woodford, Michael. 1990. "Public Debt as Private Liquidity." *American Economic Review*, 80(2): 382–388.

A Derivation of Simple Model

A.1 Households' Problem

In this section, we specify and solve the household's problem in the general case of income received in all periods and taxes paid in all periods. For household i, their objective function and budget constraints are given below:

$$\max_{C_t(i), C_{t+1}(i), C_{t+2}(i)} \mathbb{E}_t \left\{ \log\left(C_t\left(i\right)\right) + \beta \log\left(C_{t+1}\left(i\right)\right) + \beta^2 \log\left(C_{t+2}\left(i\right)\right) \right\} \tag{A.1}$$

$$\text{s.t.} \quad C_t\left(i\right) = w_t L_t\left(i\right) - T_t^y + B_t\left(i\right) \tag{A.2}$$

$$C_{t+1}\left(i\right) = Z_t + w_{t+1} L_{t+1}\left(i\right) - T_{t+1}^m + B_{t+1}\left(i\right) - \frac{\left(1+i_t\right)}{\Pi_{t+1}} B_t\left(i\right) \tag{A.3}$$

$$C_{t+2}\left(i\right) = w_{t+2} L_{t+2}\left(i\right) - T_{t+2}^o - \frac{\left(1+i_{t+1}\right)}{\Pi_{t+2}} B_{t+1}\left(i\right) \tag{A.4}$$

$$B_{t+j}\left(i\right) \leq \mathbb{E}_{t+j}\left(1 + r_{t+j+1}\right) D_{t+j} \quad \text{for } j = 0, 1 \tag{A.5}$$

where the household i has exogenous labor supply endowments in each period of life, D_{t+j} is an exogenous collateral constraint, and T_{t+j} are lump sum taxes imposed by the government. We allow taxes to differ across household types and taxes to change over time.

We restrict ourselves to cases in which the collateral constraint is binding in the first period of life and possibly binding in the second period of life. In particular, we will assume two types of households - a household that has sufficiently low labor endowment in its middle period of life and remains credit constrained, and a household that has sufficiently high labor endowment in its middle period of life and is unconstrained. For the former, borrowing in the young and middle generations is determined by the binding collateral constraints. For the latter, borrowing is determined by the collateral constraint only while young; in the middle-generation, an Euler equation determines the optimal level of saving:

$$\frac{1}{C_t^{m,h}} = \beta \mathbb{E}_t \frac{1+i_t}{\Pi_{t+1} C_{t+1}^{o,h}} \tag{A.6}$$

Let L^y be the labor endowment for young generation, $L^{m,l}$ be the labor endowment for the poor middle-generation household, $L^{m,h}$ the labor endowment for the wealthy middle-generation household, and L^o the labor endowment in the last period. We adopt the normalization that $L^y + \eta_s L^{m,l} + \left(1-\eta_s\right) L^{m,h} + L^o = 1$. The budget constraints for each type of household alive at any point in time is given below:

$$C_t^y = \alpha Y_t \frac{L^y}{L_t^{flex}} - T_t^y + \mathbb{E}_t \Pi_{t+1} \frac{D_t}{1+i_t} \tag{A.7}$$

$$C_t^{m,l} = \alpha Y_t \frac{L^{m,l}}{L_t^{flex}} + (1-\alpha) Y_t - T_t^m - D_{t-1} + \mathbb{E}_t \Pi_{t+1} \frac{D_t}{1+i_t} \tag{A.8}$$

$$C_t^{m,h} = \alpha Y_t \frac{L^{m,h}}{L_t^{flex}} + (1-\alpha) Y_t - T_t^m - D_{t-1} - B_t^{m,h} \tag{A.9}$$

$$C_t^{o,l} = \alpha Y_t \frac{L^o}{L_t^{flex}} - T_t^o - D_{t-1} \tag{A.10}$$

$$C_t^{o,h} = \alpha Y_t \frac{L^o}{L_t^{flex}} - T_t^o + B_{t-1}^{m,h} \frac{1+i_{t-1}}{\Pi_t} \tag{A.11}$$

where T_t^i are lump sum taxes per capita and Y_t is output per middle-generation household.[44] Aggregate consumption in this economy is given by the following expression:

$$C_t = N_t C_t^y + N_{t-1} \left(\eta_s C_t^{m,l} + (1-\eta_s) C_t^{m,h} \right) + N_{t-2} \left(\eta_s C_t^{o,l} + (1-\eta_s) C_t^{o,h} \right)$$

A.2 Firms' Problem, Labor Supply and Wage Determination

In this section, we specify the firm's problem in the baseline case with no capital accumulation. Firms choose labor to maximize profits subject to a standard decreasing returns to scale production function, taking wages as given:

$$Z_t = \max_{L_t} P_t Y_t - W_t L_t^d \tag{A.12}$$

$$\text{s.t.} \quad Y_t = A_t \left(L_t^d \right)^{\alpha} \tag{A.13}$$

where L_t^d is firm's labor demand. Firms' labor demand is determined by equating the real wage to the marginal product of labor:

$$\frac{W_t}{P_t} = \alpha A_t \left(L_t^d \right)^{\alpha - 1} \tag{A.14}$$

Each middle-generation household operates a firm and collects profits from its operation. The total measure of firms in the economy is N_{t-1}, and therefore grows with the total population. All firms are identical sharing the same labor share parameter α.

Labor supply is exogenous and fixed over a household's lifetime. When population is constant ($g = 0$), then labor supply is constant and can be normalized to unity. In the absence of downward nominal wage rigidity, the real wage equalizes labor supply to labor demand:

$$\left(N_t L^y + N_{t-1} \left(\eta_s L^{m,l} + (1-\eta_s) L^{m,h} \right) + N_{t-2} L^o \right) = N_{t-1} L_t^{flex} \tag{A.15}$$

[44] Output is not expressed in per capita terms to avoid a proliferation of population growth rate terms. In this economy, aggregate output is $N_{t-1} Y_t$ while the total population is $N_t + N_{t-1} + N_{t-2}$.

where w_t^{flex} defines the market-clearing real wage.

In the presence of downward nominal wage rigidity, the real wage may exceed the market clearing real wage. In this case, labor is rationed with a proportional reduction in labor employed across all households (i.e. if total labor demand is 10% below the full-employment level, then labor falls 10% for all cohorts).

We assume that nominal wages are downwardly rigid implying that real wages exceed the market-clearing level in the presence of deflation. The process determining the real wage is given below:

$$W_t = \max\left\{\tilde{W}_t, P_t w_t^{flex}\right\} \text{ where } \tilde{W}_t = \gamma W_{t-1} + (1-\gamma) P_t w_t^{flex} \tag{A.16}$$

A.3 Monetary and Fiscal Policy

Monetary and fiscal policy are straightforward. We assume a monetary policy rule of the following form:

$$1 + i_t = \max\left(1, (1+i^*)\left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi}\right) \tag{A.17}$$

where i^* is a the targeted natural rate and Π^* is the central bank's gross inflation target. If the central bank has the correct natural rate target i^*, then inflation is stabilized at $\Pi = 1$ in steady state.

Taxation is determined by the government's budget constraint and exogenous processes for government spending, the public debt, and taxation of young households. We typically assume that the ratio of taxes between old and middle-generation households satisfies the following rule:

$$T_t^o = \beta \frac{1 + i_{t-1}}{\Pi_t} T_t^m \tag{A.18}$$

In steady state, this fiscal rule ensures that changes in taxation have no effect on loan supply. We consider exceptions to this fiscal rule where taxes are levied only on old or middle-aged households respectively. The government's budget constraint together with the fiscal rule determines T_t^m and T_t^o in response to the other exogenous fiscal processes:

$$B_t^g + T_t^y (1 + g_t) + T_t^m + \frac{1}{1 + g_{t-1}} T_t^o = G_t + \frac{1}{1 + g_{t-1}} \frac{1 + i_{t-1}}{\Pi_t} B_{t-1}^g \tag{A.19}$$

where all fiscal variables are all normalized in terms of per middle-generation quantities.

A.4 Market Clearing and Equilibrium

Asset market clearing requires that total lending from savers equals total borrowing from credit constrained young households and poor middle-generation households. This condition is given below:

$$(1-\eta_s)\, N_{t-1} B_t^{m,h} = N_t \frac{D_t}{1+r_t} + \eta_s N_{t-1} \frac{D_t}{1+r_t} \tag{A.20}$$

$$(1-\eta_s)\, B_t^{m,h} = (1+g_t+\eta_s)\, \frac{D_t}{1+r_t} \tag{A.21}$$

It can be verified that asset market clearing implies that aggregate consumption equals aggregate output less aggregate government purchases:

$$C_t = N_{t-1} Y_t - (N_t + N_{t-1} + N_{t-2})\, G_t \tag{A.22}$$

A competitive equilibrium is a set of aggregate allocations $\left\{Y_t, C_t^{m,h}, C_t^{o,h}, B_t^{m,h}, L_t^{flex}, T_t^m, T_t^o\right\}_{t=0}^{\infty}$, price processes $\left\{i_t, \Pi_t, w_t, w_t^{flex}\right\}_{t=0}^{\infty}$, exogenous processes $\{G_t, g_t, D_t, T_t^y, B_t^g\}_{t=0}^{\infty}$ and initial values of household saving, nominal interest rate, real wage, and the public debt $\left\{B_{-1}^{m,h}, i_{-1}, w_{-1}, B_{-1}\right\}$ that jointly satisfy:

1. Household Euler equation ($A.6$)
2. Household budget constraints ($A.9$) and ($A.11$)
3. Asset market clearing ($A.21$)
4. Fiscal policy rule ($A.18$)
5. Government budget constraint ($A.19$)
6. Monetary policy rule ($A.17$)
7. Full-employment labor supply ($A.15$)
8. Full-employment wage rate: $w_t^{flex} = \alpha A_t \left(L_t^{flex}\right)^{\alpha-1}$
9. Labor demand condition: $w_t = \alpha A_t \left(\frac{Y_t}{A_t}\right)^{\frac{\alpha-1}{\alpha}}$
10. Wage process: $w_t = \max\left\{\tilde{w}_t, w_t^{flex}\right\}$ where $\tilde{w}_t = \gamma \frac{w_{t-1}}{\Pi_t} + (1-\gamma) w_t^{flex}$

B Linearization and Solution

In this section, we detail the linearization and general solution to the model without capital but with income received in all periods. For simplicity, we do not consider the effect of population growth shocks which greatly complicate the linearization and the computation of analytical solutions.

The generalized model with income received in all three periods and credit constrained middle-generation households can be summarized by the following linearized AD curve and linearized AS curve.

$$i_t = E_t \pi_{t+1} - s_y \left(y_t - g_t\right) + \left(1 - s_w\right) E_t \left(y_{t+1} - g_{t+1}\right) + s_w d_t + s_d d_{t-1} \tag{B.1}$$

$$y_t = \gamma_w y_{t-1} + \gamma_w \frac{\alpha}{1-\alpha} \pi_t \tag{B.2}$$

where various coefficients are given in terms of their steady state values.

$$\begin{aligned}
\gamma_w &= \frac{\gamma}{\bar{\pi}} \\
s_y &= \frac{\bar{Y}_{m,h}}{\bar{Y}_{m,h} - \bar{D}} \\
s_d &= \frac{\bar{D}}{\bar{Y}_{m,h} - \bar{D}} \\
s_w &= \frac{1+\beta}{\beta} \frac{\left(1 + \bar{g} + \eta_s\right) \bar{D}}{\bar{i}/\bar{\pi} \left(\bar{Y}_{m,h} - \bar{D}\right)}
\end{aligned}$$

The exogenous shocks are the collateral shock d_t and the government spending shock g_t, which means that a solution to this linear system takes the form:

$$y_t = \beta_y y_{t-1} + \beta_g g_t + \beta_d d_t + \beta_{d,l} d_{t-1} \tag{B.3}$$

$$\pi_t = \alpha_y y_{t-1} + \alpha_g g_t + \alpha_d d_t + \alpha_{d,l} d_{t-1} \tag{B.4}$$

Solving by the method of undetermined coefficients, we obtain the following expressions for the coefficients that determine equilibrium output and inflation in response to collateral and government spending shocks.

$$\beta_y = 0 \tag{B.5}$$

$$\alpha_y = -\frac{1-\alpha}{\alpha} \tag{B.6}$$

$$\beta_{d,l} = \frac{s_d}{s_y + \frac{1-\alpha}{\alpha}} \tag{B.7}$$

$$\alpha_{d,l} = \frac{1-\alpha}{\gamma_w \alpha} \beta_{d,l} \tag{B.8}$$

$$\beta_d = \frac{s_w + \beta_{d,l} \left(\frac{1-\alpha}{\gamma_w \alpha} + \left(1 - s_w\right)\right)}{s_y + \left(1 - s_w\right) \rho_d + \frac{1-\alpha}{\alpha} \left(1 - 1/\gamma_w \rho_d\right)} \tag{B.9}$$

$$\alpha_d = \frac{1-\alpha}{\gamma_w \alpha} \beta_d \tag{B.10}$$

$$\beta_g = \frac{s_y + \left(1 - s_w\right) \rho_g}{s_y + \left(1 - s_w\right) \rho_g + \frac{1-\alpha}{\alpha} \left(1 - 1/\gamma_w \rho_g\right)} \tag{B.11}$$

$$\alpha_g = \frac{1-\alpha}{\gamma_w \alpha} \beta_g \tag{B.12}$$

By substituting ($B.2$) into ($B.1$), we can obtain a first-order difference equation in output. This forward-looking difference equation implies that inflation and output will be determinate if and only if the following condition obtains:

$$s_y - (1 - s_w) > \frac{1-\alpha}{\alpha}\frac{1-\gamma_w}{\gamma_w}$$

When $s_w = 1$, this condition is the same determinacy condition as discussed in the main text. When the above condition holds, there is a unique rational expectations equilibrium in the deflation steady state. The left hand side is always positive, so in the case of perfect price rigidity (i.e. $\gamma_w = 1$), this condition is satisfied and the deflation steady state is locally unique.

C Properties of Secular Stagnation Equilibrium

Here we provide a formal proof for various properties of the secular stagnation equilibrium described in the body of the text.

Proposition 1. *If $\gamma > 0$, $\Pi^* = 1$, and $i^* = r^f < 0$, then there exists a unique determinate secular stagnation equilibrium.*

Proof. Under the assumptions of the proposition, the inflation rate at which the zero lower bound binds given in equation (27) is strictly greater than unity. Let Y_{AD} denote the level of output implied by the aggregate demand relation and Y_{AS} denote the level of output implied by the aggregate supply relation. For gross inflation rates less than unity, Y_{AD} and Y_{AS} are given by:

$$Y_{AD} = D + \psi\Pi \tag{C.1}$$

$$Y_{AS} = \left(\frac{1-\frac{\gamma}{\Pi}}{1-\gamma}\right)^{\frac{\alpha}{1-\alpha}} Y^f \tag{C.2}$$

where $\psi = \frac{1+\beta}{\beta}(1+g)\, D > 0$. The AD curve is upward sloping because $\Pi < 1 < \Pi_{kink}$ under our assumptions and, therefore, the zero lower bound binds.

When $\Pi = \gamma$, $Y_{AD} > Y_{AS} = 0$. When $\Pi = 1$, the real interest rate equals $\Pi^{-1} = 1 > r^f$. Thus, when $\Pi = 1$, $Y_{AD} < Y^f$. Furthermore, from the equations above, when $\Pi = 1$, $Y_{AS} = Y^f$. Therefore, it must be the case that $Y_{AD} < Y_{AS}$ when $\Pi = 1$. Since the AS and AD curve are both continuous functions of inflation, it must be the case that there exists a Π_{ss} at which $Y_{AD} = Y_{AS}$.

To establish uniqueness, we first assume that their exist multiple distinct values of Π_{ss} at which $Y_{AD} = Y_{AS}$. In inflation-output space (output on the x-axis), the AS curve lies above the AD curve when inflation equals γ and the AS curve lies below the AD curve for inflation at unity - see equation (24). Thus, if multiple steady states exist, given that AS is a continuous function, there must exist at least three distinct points at which the AS and AD curve intersect.

At the first intersection point, the slope of AS curve crosses the AD line from above and, therefore, at the second intersection the AS curve crosses the AD curve from below. Since the AD curve is a line, the AS curve as a function of output is locally convex in this region. Similarly, between the second and third intersection, the AS curve is locally concave. Thus, given an increase in Y, the AS curve must first have a positive second derivative followed by a negative second derivative.

We compute the second derivative of inflation with respect output of the AS curve and derive the following expression:

$$\frac{d^2\Pi}{dY^2} = G(Y)\left((1+\phi)(1-\gamma)\left(\frac{Y}{Y^f}\right)^{\phi} + (\phi - 1)\right) \tag{C.3}$$

$$G(Y) = \frac{\phi\gamma(1-\gamma)\left(\frac{Y}{Y^f}\right)^{\phi}}{Y^2\left(1-(1-\gamma)\left(\frac{Y}{Y^f}\right)^{\phi}\right)} \tag{C.4}$$

$$\phi = \frac{1-\alpha}{\alpha} \tag{C.5}$$

As can be seen, over the region considered, the function $G(Y)$ is positive and, therefore, the convexity of the AS curve is determined by the second term. This term may be negative if $\phi < 1$, but this expression is increasing in Y between 0 and Y_f. Therefore, the second derivative cannot switch signs from positive to negative. Thus, we have derived a contradiction by assuming multiple steady states. Therefore, there must exist a unique intersection point.

As established before, it must be the case that the AS curve has a lower slope than the AD curve at the point of intersection. The slope of the AS curve is:

$$\frac{d\Pi}{dY} = \frac{1-\alpha}{\alpha}\frac{1}{\gamma}\frac{\Pi}{Y}(\Pi - \gamma) \tag{C.6}$$

If the slope of the AS curve is less than the slope of the AD curve at the intersection point, then it must be the case that:

$$\begin{aligned}
\frac{1-\alpha}{\alpha}\frac{\Pi}{Y}\left(\frac{\Pi}{\gamma}-1\right) &< \psi^{-1}\\
\frac{1-\alpha}{\alpha}\frac{\psi\Pi}{Y}\left(\frac{\Pi}{\gamma}-1\right) &< 1\\
\frac{1-\alpha}{\alpha}\frac{Y-D}{Y}\left(\frac{\Pi}{\gamma}-1\right) &< 1\\
s_y\frac{\alpha}{1-\alpha}+1 &> \frac{\Pi}{\gamma}\\
\frac{\gamma}{\Pi}\left(s_y\frac{\alpha}{1-\alpha}+1\right) &> 1
\end{aligned}$$

The last inequality here is precisely the condition for determinacy discussed in Section 5. Thus, the unique secular stagnation steady state is always determinate as required. $\square$

D Calvo Pricing

In this section, we modify the aggregate supply block of our model to consider product market frictions instead of downward nominal wage rigidity. As in our baseline model, we assume that middle-generation households supply a constant level of labor $\bar{L}$. However, wages adjust frictionlessly to ensure that labor is fully employed in all periods.

Monopolistically competitive firms produce a differentiated good l and set nominal price periodically. Household consume a Dixit-Stiglitz aggregate of these differentiated goods implying that each firm faces the following demand schedule:

$$y_t(l) = Y_t \left(\frac{p_t(l)}{P_t} \right)^{-\theta} \tag{D.1}$$

$$P_t = \left(\int p_t^{1-\theta} dl \right)^{\frac{1}{1-\theta}} \tag{D.2}$$

where θ is the elasticity of substitution in the Dixit-Stiglitz aggregator and P_t is the price level of the consumption bundle consumed by households. Production only depends on labor and labor market clearing requires total labor demand to equal labor supply:

$$y_t(l) = L_t(l) \tag{D.3}$$

$$\bar{L} = \int L_t(l) \, dl \tag{D.4}$$

Combining labor market clearing with the demand for each product ($D.3$), we can derive an expression for output in terms of exogenous labor supply and a term that reflects losses due to misallocation from pricing frictions:

$$Y_t = \frac{\bar{L}}{\Delta_t} \tag{D.5}$$

$$\Delta_t = \int \left(\frac{p_t(l)}{P_t} \right)^{-\theta} dl \tag{D.6}$$

Under Calvo pricing, firms are periodically able to reset their prices and will choose a single optimal reset price irrespective of the time since their last price change. Under the Calvo assumption, we can dynamic expressions for inflation and the misallocation term Δ_t in terms of the reset price p_t^*:

$$1 = \chi \Pi_t^{\theta-1} + (1-\chi) \left(\frac{p_t^*}{P_t} \right)^{1-\theta} \tag{D.7}$$

$$\Delta_t = \chi \Pi_t^{\theta} \Delta_{t-1} + (1-\chi) \left(\frac{p_t^*}{P_t} \right)^{-\theta} \tag{D.8}$$

where χ is the Calvo parameter - the fraction of firms that do not adjust prices in the current period. Equations ($D.5$), ($D.7$), and ($D.8$) collectively define the aggregate supply block of the model with monopolistic competition and price friction. The IS curve and monetary policy rule close the model.

In steady state, we can derive the long-run Phillips curve by coming the steady state cases of equations ($D.5$), ($D.7$), and ($D.8$). The steady state AS curve is given below:

$$Y = \bar{L}\frac{1-\chi\Pi^{\theta}}{1-\chi}\left(\frac{1-\chi}{1-\chi\Pi^{\theta-1}}\right)^{\frac{\theta}{\theta-1}}$$

E Incorporating Money

In this section, we extend our baseline model to explicitly introduce a role for money and a money demand function. Households now have preferences for real money balances to capture the value of money in easing transactions frictions. For simplicity, we assume that households only hold money in the middle-period of life and utility over real money balances are separable. We also assume that their exists a level of real money balances $\bar{m}$ at which households are satiated in money - that is $v'(\bar{m}) = 0$.

We specify and characterize the household's problem in the case of income received in the middle-period only and taxes paid in all periods. For household i, their objective function and budget constraints are given below:

$$\max_{C_t(i),C_{t+1}(i),M_{t+1}(i),C_{t+2}(i)} \mathbb{E}_t\left\{\log\left(C_t\left(i\right)\right)+\beta\log\left(C_{t+1}\left(i\right)\right)+\beta v\left(M_{t+1}\left(i\right)\right)+\beta^2\log\left(C_{t+2}\left(i\right)\right)\right\} \tag{E.1}$$

$$\text{s.t.} \quad C_t\left(i\right) = B_t\left(i\right) - T_t^y \tag{E.2}$$

$$C_{t+1}\left(i\right) = Y_{t+1} - T_{t+1}^m + B_{t+1}\left(i\right) - M_{t+1}\left(i\right) - \frac{\left(1+i_t\right)}{\Pi_{t+1}}B_t\left(i\right) \tag{E.3}$$

$$C_{t+2}\left(i\right) = \frac{1}{\Pi_{t+1}}M_{t+1}\left(i\right) - T_{t+2}^o - \frac{\left(1+i_{t+1}\right)}{\Pi_{t+2}}B_{t+1}\left(i\right) \tag{E.4}$$

$$B_{t+j}\left(i\right) \leq \mathbb{E}_{t+j}\left(1+r_{t+j+1}\right)D_{t+j} \quad \text{for } j=0,1 \tag{E.5}$$

where $M_{t+1}(i)$ are real money balances demanded by household i. Money earns zero interest and carries a liquidity premium on bonds away from the zero lower bound. The household's money demand condition is given below:

$$C_{t+1}\left(i\right)v'\left(M_{t+1}\left(i\right)\right) = \frac{i_{t+1}}{1+i_{t+1}} \tag{E.6}$$

The above expression implicitly defines a money-demand equation. The given monetary policy rule determines real money balances via ($E.6$). Given a representative middle generation cohort and given that only the middle-generation demands money, we can drop the i and money demand per middle-generation household is:

$$M_t = v'^{-1}\left(\frac{i_t}{1+i_t}\frac{1}{C_t^m}\right) \tag{E.7}$$

The issuance of money by the central bank modifies the government's budget constraint in ($A.19$). The government's consolidated budget constraint expressed in real terms is given below:

$$B_t^g + M_t + T_t^y (1 + g_t) + T_t^m + \frac{1}{1 + g_{t-1}} T_t^o = G_t + \frac{1}{1 + g_{t-1}} \left(\frac{1 + i_{t-1}}{\Pi_t} B_{t-1}^g + \frac{1}{\Pi_t} M_{t-1} \right) \tag{E.8}$$

We assume a fiscal policy that adjust taxes T_t^y, T_t^m, and T_t^o to keep the government's consolidated liabilities, $M_t + B_t^g$ at some constant target level. In particular, this means that, in periods of deflation, the nominal stock of government liabilities is being reduced in proportion to the fall in the price level. In the steady state of a stagnation equilibrium featuring a constant rate of deflation, nominal government liabilities are contracting at the rate of deflation. Under a fiscal policy that keeps real government liabilities constant, the presence of money does not materially alter our conclusions.

F Productivity Growth and Hysteresis

In this section, we extend the baseline model to include trend productivity growth and offer a simple extension to model hysteresis - where output gaps feedback onto the productivity growth process. The extension of the model to include productivity growth does not greatly alter the basic features of the model but will allow the model to better match the dynamics of real GDP per capita. The aggregate demand block is still summarized by an asset market clearing that relates middle-generation income and the real interest rate:

$$1 + r_t = \frac{1 + \beta}{\beta} \frac{(1 + g_t) D_t}{Y_t^m - D_{t-1}} \tag{F.1}$$

$$= \frac{1 + \beta}{\beta} \frac{(1 + g_t) \tilde{D}_t}{\tilde{Y}_t^m - \tilde{D}_{t-1} \frac{A_{t-1}}{A_t}} \tag{F.2}$$

where $\tilde{X}_t = X_t / A_t$ are detrended variables. So long as the collateral constraint grows at the same rate as productivity growth, their exists a balanced growth path with a constant real interest rate in steady state and quantities growing at the rate of productivity growth. Relative to the AD curve in the baseline model, the only difference is that higher productivity growth increases saving by lowering the value of debt incurred when young.

Trend productivity growth also impacts the wage norm as the flexible price real wage rises over time. Now, deflation must exceed the rate of productivity growth for the wage norm to bind. More generally, if nominal wages are indexed to the inflation target, the shortfall of inflation below target must exceed the growth rate of productivity. The wage norm indexed to inflation is given below along with the flexible-price real wage:

$$W_t = \max \left\{ \gamma \Pi^* W_{t-1} + (1 - \gamma) P_t w_t^{flex}, P_t w_t^{flex} \right\} \tag{F.3}$$

$$w_t^{flex} = \alpha A_t \bar{L}^{\alpha - 1} \tag{F.4}$$

Real wages and output can be detrended by productivity growth to obtain stationary variables. Trend stationary real wages are given by the following expression:

$$\tilde{w}_t = \max\left\{\gamma\Pi^* \frac{\tilde{w}_{t-1}}{\Pi_t}\frac{A_{t-1}}{A_t} + (1-\gamma)\,\tilde{w}^{flex}, \tilde{w}^{flex}\right\} \tag{F.5}$$

where $\tilde{w}_t$ are detrended real wages and $\tilde{w}^{flex} = \alpha\bar{L}^{\alpha-1}$. In steady state, the wage norm binds when $\Pi^* > \bar{\Pi}\mu$ where μ is the steady state growth rate of productivity. The AS curve can be derived by substituting the following expressions for output and the full-employment level of output:

$$\tilde{w}_t = \alpha\tilde{Y}_t^{\frac{\alpha-1}{\alpha}} \tag{F.6}$$

$$\tilde{w}^{flex} = \alpha\tilde{Y}_{fe}^{\frac{\alpha-1}{\alpha}} \tag{F.7}$$

Equations ($F.2$), ($F.5$), and ($F.6$) along with the Fisher relation and the monetary policy rule jointly determine $\left\{\tilde{Y}_t, \tilde{w}_t, r_t, \Pi_t, i_t\right\}$.[45]

The modified equilibrium conditions presented in this section have simply taken productivity growth as an exogenous process. One possibility is that prolonged output gaps feedback into slower productivity growth. Productivity growth could be related to the output gap simply by positing a simple feedback process. This feedback process represents a reduced form mechanism whereby prolonged output gaps reduced productivity growth by, for example, reducing investment, technology adoption, expenditures in public or private research and development, or limiting the degree of firm entry. We posit the following feedback rule:

$$\frac{A_t}{A_{t-1}} = \mu_0\left(\frac{\tilde{Y}_t}{\tilde{Y}_{fe}}\right)^{\kappa}$$

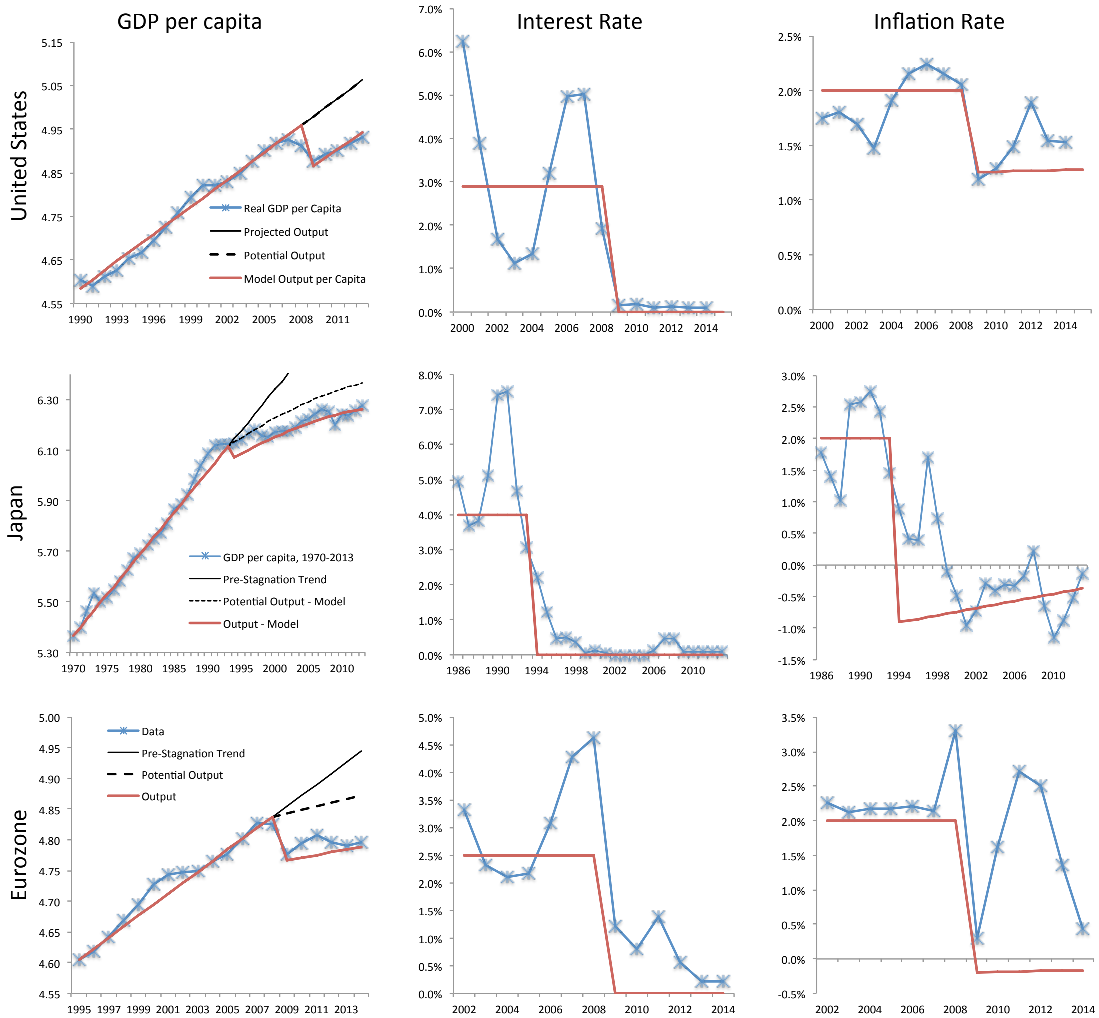
where κ determines the strength of the hysteresis effect.

G Quantitative Calibration: US, Europe and Japan

The simple three period OLG model captures the salient features of secular stagnation: persistently low levels of inflation and interest rates and below trend output. Though our model is obviously highly stylized, we still think it is of value to explicitly parameterize it and examine its capacity to explain recent stagnation episodes. Figure 9 display the key series whose behavior our theory is trying to explain. In all these episodes, we have witnessed a drop in the short-term nominal interest rate to close to zero and a decline in inflation below the implicit inflation target of the central bank. The size of the fall in inflation has varied. Japan has experience outright deflation, but in US and Europe, deflation was very short-lived while inflation has remained persistently below the target of the central banks. The size of the output gap is more controversial. Here, we

[45] Income for the middle period household $\tilde{Y}_t^m$ is assumed to be a constant fraction of total income.

Figure 9: Data v. model transition paths: US, Japan, Eurozone



calibrate the model to recent estimates, but, conceptually, even without any remaining output gap, low natural rates of interest pose an ongoing challenge for monetary policy.

To evaluate the capacity of our model to match the behavior of GDP per capita, interest rates, and inflation, we choose parameters to target steady state output gaps and deviations of inflation from target for each region. As discussed in the previous section (Appendix F), we modify the equilibrium conditions of our model to allow for trend productivity growth and wage indexation to the inflation target. In the case of Japan and the Eurozone, we assume that trend productivity growth fell at the onset of secular stagnation to fit the slower trend rate of GDP growth in each region. We log-linearize the model around the secular stagnation steady state, and plot transition paths for output per capita, nominal interest rates, and inflation in response to a shock to the collateral constraint in each region. Given the three period OLG structure, each period is taken to be 20 years.

G.1 US Calibration

Table 1 shows parameters annualized for the US (hence in computing steady state, these values need to be converted to 20 years). We pick an inflation target of 2 percent per year consistent with the Federal Reserve's implicit inflation target. Moreover, we assume that the wage norm is indexed at the inflation rate times the growth rate of productivity. In this case, the wage norm binds whenever inflation falls below target. We assume trend productivity growth of 2.1% per year to match average GDP per capita growth from 1990-2007. The rate of time preference β is set at a conventional value of 0.96. We set US population growth of 0.7% per year based upon UN population projections. We set α at 0.7 to match the labor share.

The two novel parameters to choose are the degree of wage rigidity determined by γ and the collateral constraint on the young D. We must also set an initial value for the collateral constraint as the initial point for the determining the transition path of the economy. To find realistic values of the collateral constraint, it is necessary to include other sources of demand. We do this by assuming that some income is received by the young, the government issues public debt, and the government absorbs some output through purchases that are financed by taxes levied on the middle aged. Government spending as a percentage of GDP is set at 20% and public debt as a percentage of annual GDP is set at 100%. The government budget constraint determines the level of taxes.

Given government spending, taxes, and public debt, the collateral constraint D, the income distribution between young and old, and wage rigidity parameter are set to match the following targets: an output gap of 13%, an inflation rate of 1.4%, and household debt of 100% of GDP in 2014. The output gap represents the deviation of output per capita from its 1990-2007 trendline. The inflation rate is based on the growth rate of core PCE in 2014. The household debt target is taken from the Federal Reserve Flow of Funds and is set at 100% of GDP to match the sum of loans to households and nonprofits (13.7 trillion in 2014) and loans to nonfinancial noncorporate businesses (4.4 trillion in 2014). Finally, the pre-shock level of the collateral constraint is set to match the average nominal interest of 2.9% between 2001-2007.

How reasonable are the values we set for these parameters? The implied weight on last period wages (γ multiplied by the inflation target and productivity growth) is 0.98 per year. Schmitt-Grohé and Uribe (2015) provide an authoritative overview of the evidence on downward nominal wage rigidity and estimate a very similar wage adjustment curve as we use here. The implied income distribution implies 60% of national income is received by the middle-aged while 40% of national income is received by the young. Our calibration implies an initial level of household debt of 126% of GDP which is quite close to the pre-2008 level of loans to households and small businesses of 121% of GDP pre-2008. Therefore, the collateral shock we choose is quite close in magnitude to the contraction of lending experienced during the Great Recession.

Table 8: Parameter values for US

Description	Parameter	Value
Population growth	g	0.7%
Discount rate	β	0.96
Labor share	α	0.7
Inflation target	Π^*	2.0%
Pre-shock collateral (% of annual GDP)	$\frac{1+g}{1+r_{ini}}\frac{D_{ini}}{Y}$	126%
Post-shock collateral (% of annual GDP)	$\frac{1+g}{1+r_{ss}}\frac{D_{ss}}{Y}$	100%
Wage adjustment	γ	0.94
Youth income share	Y_y/Y	0.4

The top row of Figure 9 plots the transition paths for GDP per capita, inflation, and the interest rate over the relevant data series. A collateral shock can indeed explain the drop in inflation and the interest rate in our model. The nominal interest rate drops to the zero lower bound on impact while the inflation rate slightly overshoots gradually rising towards its steady state value of 1.4%. Output per capita falls by less on impact than in the long-run. Therefore, the output gap is widening slightly along the transition path. Indeed, this is consistent with the behavior of US GDP per capita which is drifting slightly away from the pre-recession trendline. Broadly speaking, our fairly simple model with a shock that matches the magnitude of the contraction in lending can capture the behavior of GDP per capita, inflation, and interest rates in the US over the Great Recession.

G.2 Eurozone and Japan Calibration

With hysteresis effects, we can calibrate the model to analyze stagnation episodes in the Eurozone and Japan. As with the US calibration, we set the rate of time preference β and the labor share α to standard values. Population growth in both regions is set to zero to reflect recent population trends. The inflation target is set at 2% in both regions. The key remaining parameters are the collateral constraint D, the degree of wage rigidity γ, and the hysteresis parameter κ. For both regions, we set these parameters to match the output gap, inflation rate, and change in trend output growth. The pre-shock level of the collateral constraint is set to match the nominal interest rate prior to the stagnation episode.[46] In each case, output is normalized to unity.

In the case of Japan, we target an output gap of 10%, a rate of deflation of -0.25%, and a reduction in trend productivity growth from 3.3% to 0.7%. The last value is determined by trend GDP per capita growth from 1970-1994 and from 1994-2008 respectively. Data on real GDP per

[46]For this calibration, we set government debt/spending to zero, assume income is received only in the middle period, and do not seek to match the collateral constraint to measures of household debt in the Eurozone or Japan.

Table 9: Parameter values for Japan and Eurozone

Description	Parameter	Japan	Eurozone
Population growth	g	0%	0%
Discount rate	β	0.96	0.96
Labor share	α	0.7	0.7
Inflation target	Π^*	2.0%	2.0%
Pre-shock collateral	D_0	0.37	0.27
Post-shock collateral	D_1	0.23	0.21
Wage adjustment	γ	0.90	0.89
Hysteresis elasticity	κ	4.8	3.0

capita come from the World Economic Indicators. The output gap is based on estimates from Hausman and Wieland (2014), and inflation rate is the growth rate of Japan's CPI as reported in the OECD main economic indicators. As Table 9 shows, the implied degree of wage rigidity is low (relevant parameters are annualized). Because of hysteresis effects in our model, real wages are relatively flexible in comparison to the values found in Schmitt-Grohé and Uribe (2011). Hysteresis effects limit downward pressure on prices due to the paradox of toil, and therefore, limit the degree of deflation despite a persistent output gap.

In the case of the Eurozone, we target an output gap of 10%, an inflation rate of 0%, and a slowdown in productivity growth from 1.8% to 0.2%. The output gap is chosen to best fit the observed decline in output per capita during the Great Recession. The inflation rate reflects the growth rate of the harmonized CPI in the Eurozone in 2014. The slowdown in productivity growth is estimated by simple trendlines for GDP per capita growth from 1995-2008 and 2009-2014 respectively. The degree of wage rigidity in the Eurozone is comparable to values for Japan, and the hysteresis effects are about half as strong reflecting the more modest slowdown in trend GDP growth before and after the stagnation episode. Like the US, the output gap in the Eurozone is gradually increasing over time along the transition path consistent with the behavior of EU GDP per capita post Great Recession.

H Derivation of the Quantitative Lifecycle Model

H.1 Demographics and Labor Supply

The economy consists of a large number of households with identical utility parameters. Households enter economic maturity at age 26 (model age 1) after which they work, consume, have children, and participate in markets. Finally they die at age $26 + J - 1$ (model age J) that we take to be 81 years. Individuals have children at age 26, and the population growth rate is determined by the total fertility rate (TFR) of every family. People have a probability of dying stochastically before reaching maximum age J. The probability of surviving between age j and $j+1$ is denoted by s_j. The unconditional probability of reaching age j is denoted with a superscript s^j.[47]

The total population alive at any given time N_t is the sum of the population of the individual ages $N_{j,t}$. The population size of a given generation $N_{j,t}$ the population of the generation the previous year that has survived. But what about at model age 1? The total population of a generation entering economic majority at time t is equal to the total population of their parents when they entered economic maturity at time $t-25$, times the total fertility rate of their parents generation. Thus population evolves in the model according to the law of motion given below:

$$N_t = \sum_{j=1}^{J} N_{j,t}$$

$$N_{j+1,t+1} = s_{j,t} N_{j,t} \text{ for } j \in \{1, J\}$$

$$N_{1,t} = N_{1,t-25} * TFR_{t-25}$$

The total fertility rate along with the age at which individuals have children determines the rate of population growth rate. In a steady state, for a given total fertility rate TFR, the rate of population growth n is equal to:

$$n = TFR^{\frac{1}{25}} - 1 \tag{H.1}$$

In a steady state with rate of population growth n, each generation is $(1+n)$ times larger than the previous. Thus the total population size is can be calculated as (normalizing the population of age 1 population to 1):

$$N = \sum_{j=1}^{J} N_j \tag{H.2}$$

$$N_{j+1} = s_j \frac{N_j}{(1+n)} \text{ for } j \in \{1, J\} \tag{H.3}$$

$$N_1 = 1 \tag{H.4}$$

Each household has an identical schedule of lifetime exogenous labor productivity, or human capital, denoted by hc_j which varies by age. Household receive no wage income after retirement, which in our model occurs after age 65 (model age 40). We assume labor is supplied inelastically. Therefore, wage income at full employment is equal to the wage multiplied by the individual age specific labor productivity hc^j net of labor taxes $(1-\tau^w)$. In a secular stagnation, labor demand falls below labor supply, and labor is rationed proportionally for each cohort.

$$L_t^s = \sum_{j=1}^{J} N_{j,t} hc_j \tag{H.5}$$

[47] This can be calculated as the production on one-period survival probabilities: $s^j = \Pi_{m=1}^{j-1} s_m$.

H.2 Household's Problem

Individuals receive utility from two sources: (1) consumption, which is given by a time-separable constant elasticity of substitution (CES) utility function $u(\cdot)$ with an elasticity of intertemporal substitution parameter ρ, and (2) bequests, which are divided equally among all descendants. The bequest motive is also characterized by a CES function $v(\cdot)$ whose argument is the amount of bequests left to the descendent, denoted by x. The utility of bequests is multiplied by a parameter $\mu \geq 0$ which determines the strength of the bequest motive. Denoting consumption of household of age j at time t by $C_{j,t}$ and the discount rate by β, a household who is born at time t then maximizes their lifetime expected utility:

$$U_t = \sum_{j=1}^{J} s^j \beta^{j-1} u(C_{j,t+j-1}) + s^J \beta^{J-1} \mu v(x_{J,t+J-1}) \tag{H.6}$$

The household of age j can purchase or borrow a real asset $a_{j,t}$ at price ξ_t at time t which is used as productive capital. At time $t+1$ it will pay the return r^k_{t+1} which is the rental rate of capital, and has a resell value(net of depreciation) $(1-\delta)\xi_{t+1}$, where ξ_{t+1} is the relative price of capital in terms of the consumption good. All households, prior to the terminal period, participate in a perfectly competitive one period annuity market as in Ríos-Rull (1996). For a given cohort, the assets of individuals that die are distributed evenly to the surviving members of the cohort. Households also receive income from the pure profits from firms, denoted by $\Pi_{j,t}$[48]. Finally, the household may receive a bequest $q_{j,t}$. Bequest received are assumed to zero at all times except at age $J-25$. Bequest given are zero at all time except at age J.

The flow budget constraint of a household of age j at time t can be then be written as:

$$c_{j,t} + \xi_t a_{j+1,t+1} + TFR \cdot x_{j,t} = (1-\tau^w) w_t h c_j + \Pi_{j,t} + [r^k_t + \xi_t (1-\delta)] \left(a_{j,t} + q_{j,t} + \frac{1-s_j}{s_j} a_{j,t} \right) \tag{H.7}$$

Household may wish to borrow against future income, and face a borrowing constraint of the same form as in the three-period model[49]

$$a_{j,t} \geq \frac{D_t}{1+r_t} \tag{H.8}$$

There is one further complication with bequests. Since there is stochastic mortality, not all parents will survive to their maximum life age in order to give bequests to their children. Thus, absent any insurance markets, there would be stochastic within-generation inequality, as some individuals would receive bequests and others would not. In order to remove this channel, we assume all generations participate in a form of bequest insurance markets. At the maximum age, all

[48] We assume that these are distributed proportional to labor income.

[49] We assume that D_t grows at the rate of productivity growth and is expressed in terms of consumption goods to ensure balanced growth.

surviving members of a generation pool their optimal bequests, and divide them equally among their surviving children. Thus the relationship between bequests given (at model age J) and those received by children (at model age k) at time t is given by:

$$q_{k,t} = \frac{N_{J,t-1} x_{J,t-1} \cdot TFR_{t-1}}{N_{k,t}} \tag{H.9}$$

H.3 Final Goods Firms

There exist a continuum of final goods firms of type i of measure one that costlessly differentiate an intermediate good and resell to the representative household. The final good composite is the CES aggregate of these differentiated final goods:

$$Y_t = \left[\int_0^1 y_t^f(i)^{\frac{\theta_t - 1}{\theta_t}} di\right]^{\frac{\theta_t}{\theta_t - 1}}$$

These firms are monopolistically competitive, set prices each period, and face a demand curve that takes the following form:

$$y_t^f(i) = Y_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta_t}$$

where θ_t is a time-varying shock to the firm's market power. An increase in θ_t decreases a firm's market power and lowers equilibrium markups. Each final good producer uses y_t^m of intermediate good to produce output, according to a linear technology function $y_t^f = y_t^m$. A final good firm chooses real prices $\frac{p_t(i)}{P_t}$ and $y_t^f(i)$ to maximize real profits, subject to the production constraint:

$$\max \ \frac{p_t(i)}{P_t} y_t^f(i) - \frac{p_t^{int}}{P_t} y_t^f(i)$$

$$\text{subject to } \ y_t^f(i) = Y_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta_t}$$

where $\frac{p_t^{int}}{P_t}$ is the price of the intermediate good taken as given by the firm.

The optimality condition for the real price of the firm's good is a time-varying markup over the price of the intermediate good:

$$\frac{p_t(i)}{P_t} = \frac{\theta_t}{\theta_t - 1} \frac{p_t^{int}}{P_t}$$

The nominal price index is given by the following expression: and implies the following expression for the price of intermediate goods:

$$P_t = \left(\int p_t(i)^{1-\theta_t} di\right)^{\frac{1}{1-\theta_t}}$$

Since the price of the intermediate good is the same, all final goods firms make the same pricing decisions (no pricing frictions), and thus $p_t(i) = P_t$, thus we have:

$$\frac{p_t^{int}}{P_t} = \frac{\theta_t - 1}{\theta_t}$$

H.4 Intermediate Goods Firms

There exists a perfectly competitive intermediate goods sector that sells their production to the final goods sector at real price $\frac{p_t^{int}}{P_t}$. These firms operate a CES production function with an elasticity of substitution σ, hire labor, and rent capital.The representative intermediate good firms maximizes static real profits given the following production function:

$$\Pi_t^{int} = \max \ \frac{p_t^{int}}{P_t} Y_t - w_t L_t - r_t^k K_t \tag{H.10}$$

$$Y_t = \left(\alpha \left(A_{k,t} K_t\right)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\left(A_{l,t} L_t\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{H.11}$$

Labor productivity $A_{L,t}$ grows at each period at the rate of g_t.

The first order conditions that determine labor and capital demand are given below:

$$w_t = \frac{p_t^{int}}{P_t}(1-\alpha) A_{l,t} \left(\frac{Y_t}{L_t}\right)^{\frac{1}{\sigma}} \tag{H.12}$$

$$r_t^k = \frac{p_t^{int}}{P_t} \alpha A_{k,t} \left(\frac{Y_t}{K_t}\right)^{\frac{1}{\sigma}} \tag{H.13}$$

The risk-free real rate is related to the return on capital by a standard no arbitrage condition:

$$1 + r_t = \frac{r_{t+1}^k + (1-\delta)\xi_{t+1}}{\xi_t} \tag{H.14}$$

H.5 Relative Price of Capital Goods

Investment specific productivity is defined as the amount of capital goods that can be produced with one unit of investment. In particular, we will assume that capital goods are produced by perfectly competitive firms in an investment-specific production sector, who convert the final composite good into capital goods. These firms maximize the following profit function subject to a linear production function:

$$\Pi^K = \xi_t * K_t - Y_t^K \tag{H.15}$$

$$K_t = z_t Y_t^K \tag{H.16}$$

where ξ_t is the relative price of capital goods. Here, z_t is the productivity of the capital producing sector. The zero profit conditions mean that in equilibrium $\xi_t = \frac{1}{z_t}$. Thus the more productive is the investment goods sector, the lower the relative price of capital goods.

The aggregate capital stock evolves according to the standard law of motion:

$$K_{t+1} = (1-\delta) K_t + \frac{I_t}{\xi_t} \tag{H.17}$$

where δ is the rate of depreciation, I_t is investment, and ξ_t is the relative price of capital goods.

The government spends an exogenous G_t and may accumulate debt. The budget constraint is given by:

$$b_t = G_t + (1 + r_t)b_{t-1} - T_t \tag{H.18}$$

where the total tax bill is collected with labor income tax. For the purpose of our simulations, fiscal policy will be specified as an exogenous sequence of two variables: government debt to GDP and government spending to GDP. The wage income tax will then be endogenously determined by the model in order for the budget constraint to hold.

We have economized on notation by omitting real and nominal bonds as assets, they enter in the same way as in the simpler model so that there is both a well defined real interest rate r_t on a risk-free one period bond and a nominal interest rate i_t.

H.6 Equilibrium

The economy described above is not stationary when productivity growth $g \neq 0$ or population growth $n \neq 0$. However, since preferences are of the CRRA variety, the economy can be rewritten as stationary by applying a transformation. All cohort level variables are divided by $(1 + g)^t$, aggregate variables are divided by $(1 + g)^t(1 + n)^t$, and wages are divided by $(1 + g)^t$. In addition, we normalize the population of the initial generation to be of measure one. We now define the stationary equilibrium for this economy.

Definition 1. *A steady state equilibrium is a marginal product of capital r^k, a wage rate w, consumption values $\{C_j\}_{j=1}^{56}$, bequests given x, bequests received q, asset supply $\{a_i\}_{i=1}^{56}$, aggregate capital K, aggregate labor supply L^s, and aggregate output Y such that*

1. *Consumption and bequests maximizes (H.6) subject to (H.7) and (H.8)*
2. *Asset holdings satisfy (H.7), with a_1 given.*
3. *Bequests received equal bequests given by the surviving parents according to (H.9)*
4. *Population by age group is given by (H.2)*
5. *The human capital adjusted labor supply is given by (H.5)*
6. *Capital evolves according to the law of motion (H.17).*
7. *There are perfect factor markets, and thus (H.12) and (H.14) hold.*
8. *Production of intermediate goods satisfies (H.10).*
9. *The government satisfies budget equation (H.18)*

10. Asset markets clear, and thus

$$\sum_{j=1}^{J} N_j \xi a_j = \xi K + b$$

I Computational Method

Solving the quantitative model is implemented much along the lines as described in chapter 4 of Auerbach and Kotlikoff (1987). We begin by describing how to solve for variables in the stationary equilibrium. In solving for the stationary equilibrium, the algorithm begins with a guess for a subset of endogenous variables. For the purposes of the rest of the iteration, this subset of variables are treated as exogenous. This simplification makes the resulting system easier to solve for the endogenous variables, including the variables for which guesses were made.

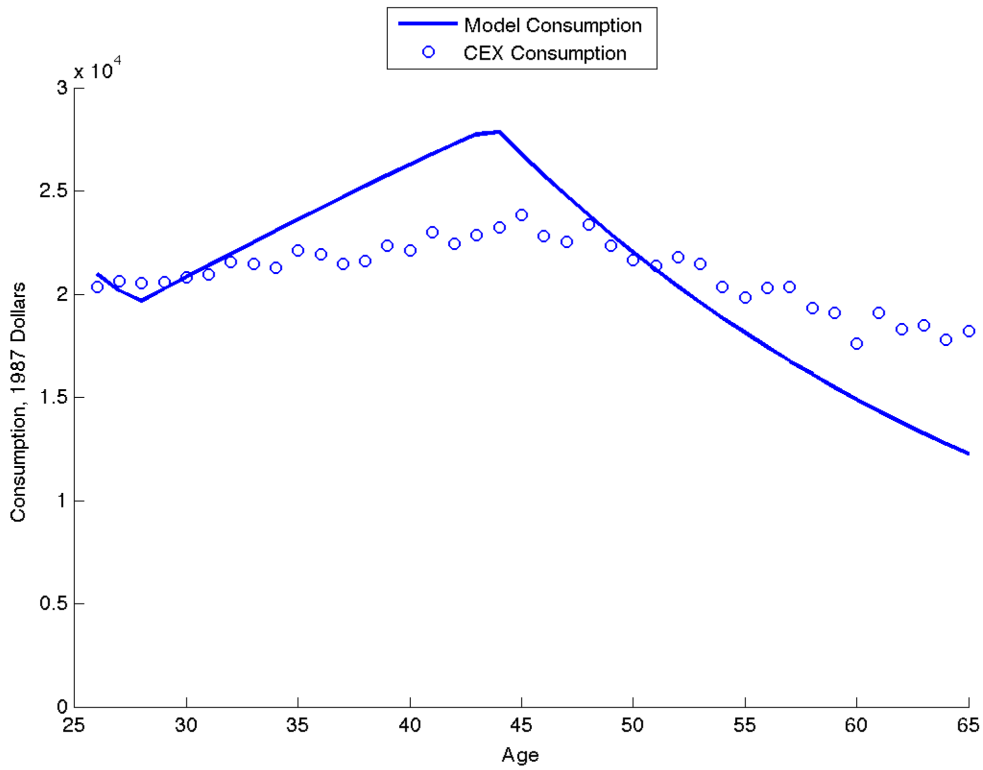
We begin with a guess for the aggregate capital stock, the wage tax, bequests received, and aggregate profits. Given the aggregate capital and labor stock, we can calculate interest rates and wages. This allows us to calculate optimal household behavior for consumption, asset holdings, and bequests given. Since the optimization problem of the household includes a debt limit, we cannot solve the household's optimal decisions analytically, as in Auerbach and Kotlikoff (1987). Instead, we use Matlab's convex optimization solver 'fmincon'.

In the steady state, the population of each age group is given by equation ($H.2$). Using the population sizes of each age group, we aggregate the optimal asset supply to obtain a new value of the aggregate capital stock. Using aggregate capital and labor, we calculate aggregate output and profits in the economy. We also update the tax rate to satisfy the government's budget constraint. Finally, we set bequests received to be equal to total bequests given by the individuals that survive to eighty-one divided by the number of children who survived to the bequest age of fifty-seven. Typically, 20-30 iterations are necessary to converge to the stationary equilibrium.

The approach used to solve for the transition path is similar to solving for the initial and final steady states of the model. We assume that the economy will be in a steady state after 150 years in the transition. In the years between the initial and final steady state, we must solve for all endogenous variables, including the optimal consumption and bequests choices for individuals of all generations. We assume the year 1 is in steady state, and that in the year 2, agents are surprised by an unexpected change in the *path* of key economic variables, such as productivity growth, total fertility rate, mortality profiles, etc. Their asset choices were made previously in period 1, and they must now adjust their consumption and saving choices to the path of these variables. At the time of the shock, households have perfect foresight.

Our algorithm now begins with a guess for the same subset of endogenous variables, but now the guess for each variable is a 150 x 1 vector for each year of the transition path. Given the vector of guesses for the capital stock, we can calculate a vector of prices and wages, and thus

Figure 10: Consumption profile in stationary equilibrium



optimally solve for consumption and bequests of each generation in the transition period. After the year 151, agents assume the economy will be in the final steady state. We then aggregate the individual asset supply decisions to get a new value for capital supply, and repeat the algorithm until convergence.

J Supplemental Figures and Tables

We perform several robustness exercises. The first alternate specification holds all variables at the levels in the main specification, however sets depreciation to be 8%. The second alternate specification holds all variables at the level in the main specification, but sets the utility elasticity of substitution equal to 1. The third alternate specification holds all parameters at the level as in the main specification, but sets the production elasticity equal to 1 (Cobb-Douglas).

Figure 11: Population pyramid in stationary equilibrium

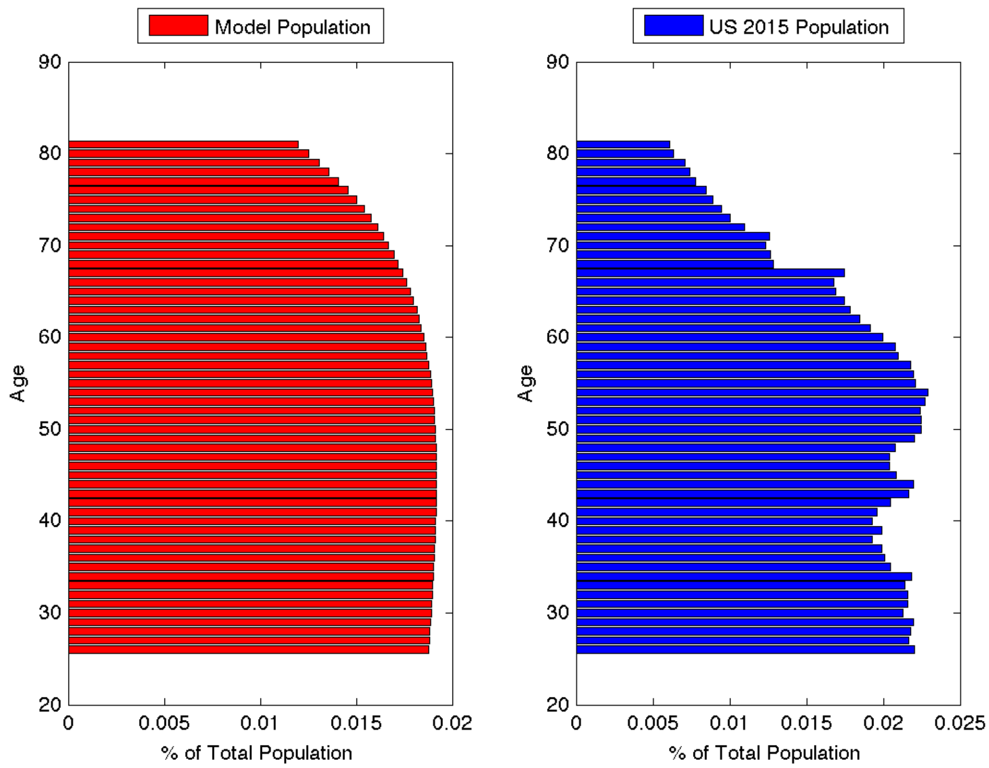


Table 10: 2015 steady state results, alternate specification 1

Moment	*Model*	*US data*
Natural rate of interest	-1.47%	-1.47%
Investment to output ratio	15.9%	15.9%
Consumer debt to output ratio	6.3%	6.3%
Labor share	65.88%	65.99%
Bequests to output	3%	3%

Table 11: Decomposition, alternate specification 1

Forcing variable	Δ in r	% of total Δ
Total interest rate change	-3.60%	100%
Mortality rate	-1.28	34%
Total fertility rate	-1.19	32%
Productivity growth	-1.67	45%
Government debt (% of GDP)	$+1.13$	-30%
Labor share	$-.45$	12%
Relative price of investment goods	-0.36	10%
Change in debt limit	$+.08$	-2%

Table 12: Raising the rate of interest to 1%, alternate specification 1

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	5.40
Government debt (% of GDP)	118%	285%
Productivity growth	0.65%	2.65%
Relative price of investment goods	1.00	2.22

Table 13: 2015 steady state results, alternate specification 2

Moment	*Model*	*US data*
Natural rate of interest	-1.47%	-1.47%
Investment to output ratio	15.9%	15.9%
Consumer debt to output ratio	6.3%	6.3%
Labor share	65.99%	65.99%
Bequests to output	3.01%	3%

Table 14: Decomposition, alternate specification 2

Forcing variable	Δ in r	% of total Δ
Total interest rate change	-2.86%	100%
Mortality rate	-1.22	43%
Total fertility rate	-1.36	47%
Productivity growth	-1.28	45%
Government debt (% of GDP)	$+1.54$	-54%
Labor share	$-.33$	12%
Relative price of investment goods	-0.30	10%
Change in debt limit	$+.08$	-3%

Table 15: Raising the rate of interest to 1%, alternate specification 2

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	4.35
Government debt (% of GDP)	118%	279%
Productivity growth	0.65%	3.29%
Relative price of investment goods	1.00	1.78

Table 16: 2015 steady state results, alternate specification 3

Moment	*Model*	*US data*
Natural rate of interest	$-.81\%$	-1.47%
Investment to output ratio	18.03%	15.9%
Consumer debt to output ratio	6.3%	6.3%
Labor share	65.40%	65.99%
Bequests to output	3%	3%

Table 17: Decomposition, alternate specification 3

Forcing variable	Δ in r	% of total Δ
Total interest rate change	-2.85%	100%
Mortality rate	-1.45	46%
Total fertility rate	-1.41	45%
Productivity growth	-1.70	54%
Government debt (% of GDP)	$+1.58$	-50%
Labor share	$-.3$	10%
Relative price of investment goods	0	0%
Change in debt limit	$+.13$	-4%

Table 18: Raising the rate of interest to 1%, alternate specification 3

Forcing variable	2015 Value	Counterfactual value
Total fertility rate	1.88	3.18
Government debt (% of GDP)	118%	208%
Productivity growth	0.65%	2.11%
Relative price of investment goods	1.00	$\nexists$