Crisis and Commitment: Inflation Credibility and the Vulnerability to Sovereign Debt Crises

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Abstract

We explore the role of inflation credibility in self-fulfilling debt crises. In particular, we propose a continuous time model of nominal debt with the potential for self-fulfilling debt crises as in Calvo (1988) and Cole and Kehoe (2000). We characterize crisis equilibria conditional on the level of commitment to low inflation. With strong commitment, which can be interpreted as joining a monetary union or issuing foreign currency debt, the environment is a version of the one studied by Cole and Kehoe. The paper contrasts this framework with one in which sovereign debt is nominal and is vulnerable to ex post devaluation. Inflation is costly, but reduces the real value of outstanding debt without the full punishment of default. In a debt crisis, a government may opt to inflate away a fraction of the real debt burden rather than explicitly default. This flexibility potentially reduces the country's exposure to self-fulfilling crises. On the other hand, the government lacks commitment not to inflate in the absence of crisis. This latter channel raises the cost of debt in tranquil periods and makes default more attractive in the event of a crisis, increasing the country's vulnerability. We characterize the interaction of these two forces. We show that there is an intermediate level of commitment that minimizes the country's exposure to rollover risk. On the other hand, low inflation credibility brings the worst of both worlds – high inflation in tranquil periods and increased vulnerability to a crisis. Weak inflationary commitment also reduces the country's equilibrium borrowing limit. These latter results shed light on the notions of original sin and debt intolerance highlighted in the empirical literature; that is, the fact that developing economies issue debt exclusive in foreign currency to international investors as well as encounter solvency issues at relatively low ratios of debt-to-GDP.

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1 Introduction

We propose a tractable, continuous-time model of self-fulfilling debt crises with nominal bonds. Our focus is on the effect of inflation credibility in determining the vulnerability to rollover risk. The option to inflate may be a powerful tool in response to a crisis, and indeed may even eliminate a crisis equilibria all together. Our framework allows us to explore when this is true, why it may fail, and implications for welfare.

We build on the canonical models of Calvo (1988) and Cole and Kehoe (2000).[1] In particular, the sovereign lacks commitment to repay debt, debt is non-contingent, and the incentive to default depends on the equilibrium interest rate, raising the possibility of multiple equilibria. We extend the Cole and Kehoe framework to nominal debt in an environment in which the government chooses the inflation rate subject to a utility cost of high inflation. The latter cost proxies for the government's commitment to low inflation in a manner reminiscent of the reputational punishment following an outright default. Letting this cost become arbitrarily large recaptures the Cole and Kehoe framework of a small open economy issuing foreign currency debt (or a small member of a monetary union facing idiosyncratic rollover risk).[2] We also recast the Cole and Kehoe framework in continuous time, which allows simple, analytical solutions to the government's value function and associated optimal policies. We maintain the Cole and Kehoe benchmark of deterministic output to focus on the uncertainty arising from the bond market.

A major finding of the analysis is that nominal debt has an ambiguous impact on the possibility of a self-fulfilling debt crisis. To provide intuition for this ambiguity, start with the Cole and Kehoe model of real bonds and a zero-one default decision. If creditors fail to roll over bonds, the government is faced with a choice of default versus repaying the principal on all outstanding debt. For large enough debt levels, default is preferable, and this may be the case even if the government is willing to service interest payments rather than default, raising the possibility of self-fulfilling debt crises.

If debt is denominated in domestic currency, the government has a third option; namely,

[1] The literature on self-fulfilling debt crises is large, some of which is surveyed and discussed in Aguiar and Amador (in progress). In addition to Calvo (1988) and Cole and Kehoe (2000), our paper is related to Da-Rocha et al. (forthcoming) which models the interplay of devaluation expectations and default in a model in which debt is denominated in foreign goods and the government chooses both a real exchange rate and a debt policy. Recent papers exploring themes involving currency denomination of debt or self-fulfilling crises include Jeanne (2011) and Roch and Uhlig (2011).

[2] In a companion paper, Aguiar et al. (2012), we discuss the fiscal externalities that arise in a currency union in the presence of limited commitment and vulnerability to self-fulfilling debt crises. In the model presented below, issuing foreign currency debt can be viewed as being a small member of a large currency union which can credibly commit to low inflation and no bailouts. Moving away from this extreme, membership in a currency union involves different inflation and debt dynamics than in the small open economy model presented here. See Aguiar et al. (2012) for details.

inflate away part of the principal and repay the rest. What is perhaps the conventional wisdom regarding debt crises is that this third option lowers the burden of repayment and eliminates the possibility of default, at least for a range of debt stocks. That is, adding another policy instrument (partial default through inflation) reduces the occurrence of outright default. However, this conclusion must be tempered by the fact that the lack of commitment to bond repayments also extends to inflation. If the commitment to low inflation is weak, then high inflation will be the government's policy even in the absence of a crisis. This drives up interest rates on debt in the non-crisis equilibrium, making default relatively attractive in all equilibria. This latter effect can generate an environment in which nominal bonds are more vulnerable to self-fulfilling runs; that is, the option for partial default makes outright default more likely.

More precisely, we establish a threshold for inflationary commitment below which an economy is more vulnerable to crises for a larger range of debt. A middle-range of inflationary commitment generates the conventional wisdom of less vulnerability. It is this level of commitment in which the economy can best approximate the state-contingent policy of low inflation in tranquil periods and high inflation in response to a liquidity crisis. Full commitment to low inflation renders nominal bonds into real bonds, recovering the Cole and Kehoe analysis.

In terms of welfare, a weak commitment to inflation is strictly dominated by issuing foreign currency (real) bonds, as the vulnerability to a crisis is greater and inflation is high in all equilibria. This rationalizes the empirical fact that emerging markets typically issue bonds to foreign investors solely in foreign currency, so-called "original sin." A moderate commitment to inflation makes nominal bonds strictly preferable for intermediate levels of debt, where the reduction in crisis vulnerability is at work. Moreover, a strong commitment to low inflation raises the limit on borrowing in the good equilibrium by keeping nominal rates low. Strong commitment to low inflation also reduces the incentive to save, as reducing debt is no longer necessary to recapture inflationary commitment in equilibrium. Countries with newly minted inflation credibility (either through a monetary union or a hard peg) often increase their borrowing in practice. Our framework captures this through the prediction that a strong commitment to low inflation reduces the incentive to lower debt levels and raises the borrowing limit in the non-crisis equilibrium.

2 Environment

2.1 Preferences and Endowment

We consider a continuous-time, small-open-economy environment. There is a single, freely-traded consumption good which has an international price normalized to 1. The economy is endowed with y units of the good each period. We consider an environment in which income is deterministic, and for simplicity assume that y is independent of time. The local currency price (relative to the world price) at time t is denoted $P_t = P(t) = P(0)e^{\int_0^t \pi(t)dt}$, where $\pi(t)$ denotes the rate of inflation at time t. To set a notational convention, we let $\boldsymbol{\pi} : [0,\infty) \to \mathbb{R}_+$ denote inflation as function of time and let $\pi(t)$ or π_t denote the evaluation of $\boldsymbol{\pi}$ at time t. When convenient, we use $\pi \in R_+$ to denote a particular inflation choice. A similar convention is used for other variables of interest, like consumption and debt.

The government has preferences over paths for aggregate consumption and domestic inflation, $x(t) = (c(t), \pi(t)) \in \mathbb{R}^2_+$, given by:

$$U = \int_0^\infty e^{-\rho t} v(x(t))dt = \int_0^\infty e^{-\rho t} \left(u\left(c(t)\right) - \psi(\pi(t))\right) dt. \tag{U}$$

Utility over consumption satisfies the usual conditions, $u' > 0, u'' < 0$, $\lim_{c\downarrow 0} u'(c) = \infty$, plus an upper bound restriction: $\lim_{c\to\infty} u(c) \leq \bar{u} < \infty$ needed for technical reasons. Power utility with a relative risk aversion coefficient greater than one satisfies these conditions.

The disutility of inflation is represented by the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\psi' > 0$ and $\psi'' \geq 0$. In the benchmark model discussed in the text, we let $\psi(\pi) = \psi_0 \pi$, $\psi_0 \geq 0$, and we restrict the choice of inflation to the interval $\pi \in [0, \bar{\pi}]$. In Appendix B, we consider a strictly convex cost of inflation, and use numerical examples to explore the robustness of the benchmark's analytical results. While we do not micro-found preferences over inflation, a natural interpretation is that ψ is a reduced-form proxy for a reputational cost to the government of inflation. A large cost represents an environment in which the government has a relative strong incentive for (or commitment to) low inflation. The reputational cost can be augmented by real distortions to a good that enters separably from tradable consumption. Allowing for inflation to reduce the (instantaneous) tradable endowment as well would pose no difficulties; for example, replacing $y(t) = y$ with $y(t) = (1 - \pi(t))y$. The cost ψ is not state contingent; in particular, the costs of inflation will be independent of the behavior of creditors, although we discuss implications of relaxing this assumption in section 4.5.

The government chooses $x = (c, \pi)$ from a compact set $X \equiv [0, \bar{c}] \times [0, \bar{\pi}]$. The upper bound on consumption $\bar{c}$ is assumed to never bind.[3] The upper bound on π will bind in the

[3]As we discuss in the next sub-section, we impose and upper bound on assets (or lower bound on debt),

benchmark case of linear cost, and as we shall see it implies a discrete choice between low (zero) inflation or high ($\bar{\pi}$) inflation. Let $\mathbf{X}$ denote admissible controls: the set of measurable functions of time $\mathbf{x} : [0, \infty) \to X$.

2.2 Bond Contracts and Budget Sets

The government can trade a nominal non-contingent bond. Let B_t denote the outstanding stock of nominal bonds, and let $b_t \equiv B_t/P_t$ denote the real value of outstanding debt. The government contracts with competitive (atomistic) risk-neutral lenders who face an opportunity cost in real terms given by the world interest rate $r^\star = \rho$. Bonds carry an instantaneous interest rate that is conditional on the outstanding stock of real debt. In particular, we consider stationary equilibria in which the government faces a time-invariant interest rate schedule $r : \overline{\Omega} \to R_+$, where $\overline{\Omega} = [b_{min}, b_{max}]$ denotes the domain of real debt permissible in equilibrium. The debt domain is characterized by a maximal debt level $b_{max} \in R_+$ above which the government cannot borrow. The value of b_{max} will be an equilibrium object. For expositional convenience, we put a lower bound on debt (or an upper bound of assets) of $b_{min} \in \mathbb{R}_-$; the analysis is not sensitive to this bound. As the government is the unique supplier of its own bonds, it understands the effects of its borrowing decisions on the cost as given by the entire function r.

The evolution of nominal debt is governed by:

$$\dot{B}(t) = P(t)(c(t) - y) + r(b(t))B(t).$$

Dividing through by $P(t)$ and using the fact that $\dot{B}/B = \dot{b}/b + \pi$ gives the dynamics for real debt:

$$\dot{b}(t) = f(b(t), x(t)) \equiv c(t) - y + (r(b(t)) - \pi(t))b(t). \tag{1}$$

We are interested in environments in which r may not be a continuous function. For technical reasons, we need to place some restrictions on the nature of these discontinuities.

Definition 1. *Given a domain $\overline{\Omega} = [b_{min}, b_{max}]$, the set $\mathcal{R}(\overline{\Omega})$ consists of functions $r : \overline{\Omega} \to \mathbb{R}_+$ such that*

(i) r is bounded and lower semi-continuous on $\overline{\Omega}$;

so an upper bound on consumption does not become an issue. The upper bound on assets is not restrictive for the analysis.

(ii) *r is such that $y - (r(b) - \bar{\pi})b \geq M > 0$ for all $b \in \overline{\Omega}$; that is, it is always feasible to have $\dot{b} = 0$ with strictly positive consumption;*

(iii) *r contains a finite number of discontinuities denoted by $b_1, b_2, ..., b_N$ with $b_{min} < b_n < b_{n+1} < b_{max}$ for all $n \in \{1, 2, .., , N-1\}$;*

(iv) *r is Lipschitz continuous on sets Ω_n for all $n \in 0, ..., N$, where $\Omega_0 \equiv (b_{min}, b_1)$; $\Omega_n \equiv (b_n, b_{n+1})$ for $n = 1, ..., N-1$; and $\Omega_N = (b_N, b_{max})$.*[4]

Denote the closure of Ω_n as $\overline{\Omega}_n$, and note that $\overline{\Omega} = \cup_{n=0}^{N} \overline{\Omega}_n$.

The debt-dynamics equation (1) implies that $b(t)$ is always continuous; however, $\dot{b} = f(b, x)$ may not be continuous in b. For $r \in \mathcal{R}(\overline{\Omega})$, continuous policies imply continuous dynamics except at finitely many points $\{b_1, ..., b_N\}$, at which the dynamics can change discretely.

For a given policy x, debt dynamics are governed by $f(b, x) = c + (r(b) - \pi)b - y$, as defined in equation (1). It will be useful to also define:

$$f_*(b, x) \equiv c - y - \pi b + \liminf_{b' \to b} r(b')b', \tag{2}$$

which represents the debt dynamics associated with a given policy under the most favorable interest rate for the government in the neighborhood of b. These alternative dynamics will be relevant at points of discontinuity in $r(b)$.

2.3 Limited Commitment

The government cannot commit to repay loans or commit to a path of inflation. At any moment, it can default and pay zero, or partially inflate away the real value of debt. As noted above, we model the cost of inflation with the loss in utility $\psi(\pi)$. We model outright default as follows. If the government fails to repay outstanding debt and interest at a point in time, it has a grace period of length δ in which to repay the bonds plus accumulated interest.During this period, it cannot issue new debt, but is also not subject to the full sanctions of default. If it repays within the grace period, the government regains access to bond markets with no additional repercussions. If the government fails to make full repayment within the grace period, it is punished by permanent loss of access to international debt markets plus a potential loss to output. We let $\underline{V}$ represent the continuation value after a default, which we assume is independent of the amount of debt at the time of default.[5]

[4] That is, for all n, there exists $K_n < \infty$ such that $r(b') - r(b'') \leq K_n |b' - b''|$ for all $(b', b'') \in \Omega_n \times \Omega_n$.

[5] For concreteness, we can define $\underline{V} = u((1-\tau)y)/\rho$ as the autarky utility, where $\tau \in [0, 1)$ represents the reduction in endowment in autarky.

We assume that $\underline{V} > u(0)/\rho$, so the country prefers default to consuming zero forever. We discuss the payoff to utilizing the grace period in section 4.1.

Modeling limited commitment in this manner has a number of advantages. First, by separating the costs of inflation from the costs of outright default, we can consider environments in which the two are treated differently by market participants. It may be the case that the equilibrium costs or "punishment" of inflation may be greater or less than the that of outright default, and the model encompasses both alternatives. For example, the high inflation of the 1970s in the US and Western Europe eroded the real value of outstanding bonds; however, the governments did not negotiate with creditors or lose access to bond markets, as typically occurs in cases of outright default. A short-coming of the analysis is we do not present a micro-founded theory of why these costs may differ in practice; we take them as primitives, and explore the consequences for debt and inflation dynamics.

Second, in practice countries can exit default status by repaying outstanding debt in full. We proxy this with a grace period, which allows the government to avoid the full punishment of default by repaying outstanding principal and interest. As we shall see, in equilibrium the government will opt for full repayment only if the payoff to doing so weakly dominates $\underline{V}$. In this manner, a grace period allows a tractable, continuous time representation such that it is feasible to repay a positive stock of debt even if creditors do not purchase new bonds.[6]As with the costs of inflation and default, we treat δ as a primitive of the environment.

3 No-Crisis Equilibria

We first characterize equilibria in which creditors can commit to (or coordinate on) rolling over debt. In particular, we assume that the government can always trade bonds at an equilibrium schedule r with no risk of a rollover crisis. There remains limited commitment on the part of the government with regard to inflation and default.

We solve the government's problem under the restriction that default (with or without subsequent repayment) is never optimal on the domain $\overline{\Omega}$. This is not restrictive in equilibrium.In particular, in the deterministic environment under consideration in this section, the equilibrium restricts debt to a domain on which it is never optimal to default.

For a given $\overline{\Omega}$; $r \in \mathcal{R}(\overline{\Omega})$; and for all $b_0 \in \overline{\Omega}$; the government's value function can be

[6] An alternative formulation is the one in He and Xiong (2012) in which each debt contract has a random maturity, which generates an explicit *iid* sequencing of creditors at any point in time. Long-maturity debt poses tractability issues in solving for an equilibrium given that the interest rate charged to new debt is a function of the inflation policy function over the bond's maturity horizon.

written as

$$V(b_0) = \max_{\mathbf{x}\in\mathbf{X}} \int_0^\infty e^{-\rho t} v(x(t))dt, \tag{P1}$$

subject to:

$$b(t) = b_0 + \int_0^t f(b(t), x(t))dt, \text{ and}$$

$$b(t) \in \overline{\Omega} \text{ for all } t.$$

Before discussing the solution to the government's problem, we define our equilibrium concept:

Definition 2. *A* Recursive Competitive Equilibrium *is an interval* $\overline{\Omega} = [b_{min}, b_{max}]$*, an interest rate schedule* $r : \overline{\Omega} \to \mathbb{R}_+$*, a consumption policy function* $C : \overline{\Omega} \to [0, \bar{c}]$*, an inflation policy function* $\Pi : \overline{\Omega} \to [0, \bar{\pi}]$*, and a value function* $V : \overline{\Omega} \to \mathbb{R}$ *such that*

(i) $r \in \mathcal{R}(\overline{\Omega})$*;*

(ii) given $(\overline{\Omega}, r)$ *and for any* $b_0 \in \overline{\Omega}$*, the policy functions combined with the law of motion* (1) *and initial debt* b_0 *generate sequences* $x(t) = (C(b(t)), \Pi(b(t)))$ *that solve the government's problem* (P1) *and deliver* $V(b_0)$ *as a value function;*

(iii) given $C(b)$ *and* $\Pi(b)$*, bond holders earn a real return* $r^\star$*, that is,* $r(b) = r^\star + \Pi(b)$ *for all* $b \in \overline{\Omega}$*; and*

(iv) $V(b_0) \geq \underline{V}$ *for all* $b \in \overline{\Omega}$*.*

The final condition imposes that default is never optimal in equilibrium. In the absence of rollover risk, there is no uncertainty and any default would be inconsistent with the lender's break-even requirement. As we shall see, condition (iv) imposes a restriction on the domain of equilibrium debt levels.[7]

We solve the government's problem using the continuous time Bellman equation. Let $H(b, q) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be defined as

$$\begin{aligned} H(b, q) &= \max_{x\in X} \{v(x) + qf(b, x)\} \\ &= \max_{(c,\pi)\in X} \{u(c) - \psi(\pi) + q(c - y + (r(b) - \pi)b)\}. \end{aligned}$$

[7] It must also be the case that the government never prefers to default and then repay within the grace period. We postpone that discussion until section 4.1.

Note that H is defined conditional on an equilibrium interest rate schedule, which we suppress in the notation. The Hamilton-Jacobi-Bellman equation is:

$$\rho V(b) - H(b, V'(b)) = 0. \tag{HJB}$$

For points of discontinuity in $r(b)$, it is also useful to define $H_*(b,q) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ as

$$\begin{aligned} H_*(b,q) &= \max_{x \in X} \{v(x) + q f_*(b,x)\} \\ &= \max_{(c,\pi)\in X} \left\{ u(c) - \psi(\pi) + q\left(c - y + \liminf_{b' \to b} (r(b') - \pi)b'\right)\right\}, \end{aligned} \tag{3}$$

which is the Bellman equation under the government's "best-case" dynamics $f_*(b,x)$. As we shall see below, this will provide an upper bound on the value function in the neighborhood of points of discontinuity in $r(b)$.

We proceed to show that the value function is the unique solution to (HJB). There are two complications. The first is that r may not be continuous, so the HJB may be discontinuous in b. The second is that the value function may not be differentiable at all points, so its derivative, $V'(b)$, may not exist. Nevertheless, the value function is the unique solution to (HJB) in the viscosity sense. We use the definition of viscosity introduced by Bressan and Hong (2007) for discontinuous dynamics adapted to our environment:

Definition 3. *For a given $\overline{\Omega}$ and $r \in \mathcal{R}(\overline{\Omega})$, a* **viscosity solution** *to (HJB) is a continuous function $w \in C^0(\overline{\Omega})$ such that for any $\varphi \in C^1(\overline{\Omega})$ we have:*

(i) If $w - \varphi$ achieves a local minimum at b, then

$$\rho w - H_*(b', \varphi'(b)) \le 0;$$

(ii) If the restriction of $w - \varphi$ to Ω_n achieves a local maximum at $b \in \Omega_n$, then

$$\rho w - H(b', \varphi'(b)) \ge 0,$$

where Ω_n is defined in definition 1;

(iii) For $b \in \{b_{min}, b_1, b_2, ..., b_{max}\}$, $\rho v(b) \ge \max_{\pi \in \{0,\bar{\pi}\}} \frac{u(y-(r(b)-\pi)b)-\psi(\pi)}{\rho}$.

We make a few remarks on these conditions. First, suppose V is differentiable at b and r is continuous at b. In this case, a local max or min of $V - \varphi$ implies $V'(b) = \varphi'(b)$. Moreover, continuity of r at b implies that $H_*(b, \varphi'(b)) = H(b, \varphi'(b))$. The first two conditions then are equivalent to the classical Bellman equation $\rho V - H(b, V'(b)) = 0$.

Now suppose that r is continuous at b, so that b is in the open set Ω_n, but V has a kink at b. As $H(b,q)$ is convex in q, a few steps show the two conditions imply that V cannot have a concave kink at b.[8] That is, V is semi-convex at b, which means the smooth portions of the function can be either concave or convex, but the non-differentiable point must be convex.[9]

Finally, suppose that r jumps discretely, up or down, at b, so $b = b_n$ for some $n \in \{1, \ldots, N\}$. To make things concrete, suppose $b_n > 0$ and thus the H_* selects the lowest interest rate: $\liminf_{b' \to b_n} r(b')$. Condition (i) then states that the value function is weakly less than if the government could carry over the low interest rate into the neighboring domain. If $b_n < 0$, the condition states the government cannot carry over a high interest rate on its assets. This provides a natural upper bound on the the value function at points of discontinuity. The government always has the option of staying put at the point of discontinuity, and thus the value function is weakly greater than the steady state value function, which is condition (iii). Note that condition (ii) only refers to the open sets on which the interest rate is continuous, and thus condition (iii) provides the relevant floor on the value function at the boundary points.

The following states that we can confine attention to the viscosity solution of (HJB):

Proposition 1. *For a given $\overline{\Omega}$ and $r \in \mathcal{R}(\overline{\Omega})$, the government's value function is the unique bounded Lipschitz-continuous viscosity solution to* (HJB).

We now characterize equilibria in the no-rollover-crisis environment. We first state that $r(b)$ takes two values, $r^\star$ and $r^\star + \bar{\pi}$, and is monotone in b.

Lemma 1. *In any no-crisis equilibrium, $r(b) \in \{0, \bar{\pi}\}$ and is non-decreasing for all $b \in \overline{\Omega}$. In particular, in any equilibrium there exists a b_π such that $r(b) = r^\star$ for $b \in [b_{min}, b_\pi]$ and $r(b) = r^\star + \bar{\pi}$ for $b \in (b_\pi, b_{max}]$.*

The intuition for monotonicity follows from the fact that the incentives to inflate increase with b. The one subtlety is that the incentives to inflate decrease with consumption, and so the result also relies on the fact that consumption is non-increasing in b, which is established below. Given that the costs of inflation are linear, there may exist a debt level in which the

[8] In particular, suppose $V(b)$ has a concave kink at $b \in \Omega_n$; that is, $V'(b^-) > V'(b^+)$. This implies that we can find φ such that $V - \varphi$ has a maximum at b, with $\varphi'(b) \in [V'(b^+), V'(b^-)]$. Using the fact that V is smooth as we approach from the left or right of b, continuity of V and r implies that $\rho V(b) = H(b, V'(b^-)) = H(b, V'(b^+))$. The fact that $V'(b^-) > V'(b^+)$ implies that $c(b^-) > c(b^+)$ and $\pi(b^-) \leq \pi(b^-)$, which in turn implies that $H_2(b, V'(b^-)) > 0 > H_2(b, V'(b^+))$. Strict convexity of $H(b,q)$ in q then implies that $H(b,q) > H(b, V'(b^-)) = \rho V(b)$ for $q \in (V'(b^+), V'(b^-))$, which violates condition (ii).

[9] More precisely, $w(b)$ is semi-concave (with linear modulus) on Ω_n if there exists $C \geq 0$ such that $w(b+h) + w(b-h) - 2w(b) \leq C|h|^2$, for all $b, h \in \mathbb{R}$ such that $[b-h, b+h] \in \Omega_i$. A function $w(b)$ is semi-convex if $-w(b)$ is semi-concave.

government is indifferent between inflation rates. Nevertheless, the requirement that $r(b)$ be lower semi-continuous implies that the point of indifference produces zero inflation.

The threshold b_π characterizes the equilibrium $r(b)$. We can define an interval which contains all possible b_π:

Definition 4. *The values $\bar{b}_\pi$, $\underline{b}_\pi$ are given by the unique solutions to:*

$$\psi_0 = u'(y - r^\star \bar{b}_\pi)\bar{b}_\pi, \qquad \textit{and} \qquad \psi_0 = u'(C_\pi(\underline{b}_\pi))\underline{b}_\pi$$

where $b \mapsto C_\pi(b) \in (0, y - r^\star b)$ is defined uniquely by[10]

$$u(y - r^\star b) - u(C_\pi(b)) + \psi_0 \bar{\pi} + u'(C_\pi(b))(C_\pi(b) - y + r^\star b) = 0. \tag{4}$$

Note that both $\bar{b}_\pi$ and $\underline{b}_\pi$ exist and are such that $y/r^\star > \bar{b}_\pi > \underline{b}_\pi > 0$. The threshold $\bar{b}_\pi$ relates to the optimal decision of the government regarding inflation when it faces the interest rate $r^\star$ and consumes $c = y - r^\star b$. As $r^\star = \rho$, there is no incentive to save or borrow if it has zero inflation and faces the interest $r^\star$. From (HJB), and using the envelope condition $-V'(b) = u'(c) = u'(y - r^\star b)$, low inflation is optimal as long as $u'(y - r^\star b)\bar{\pi} b - \psi_0 \bar{\pi} \geq 0$. For $b > \bar{b}_\pi$, this condition is violated. The threshold $\underline{b}_\pi$ and the associated function C_π relate to the solution of (HJB) when the interest rate is $r^\star + \bar{\pi}$. In particular, as discussed below, $C_\pi(b_\pi)$ denotes the optimal consumption assuming high inflation in the neighborhood above b_π, and the condition defining $\underline{b}_\pi$ ensures that optimal consumption is consistent with high inflation.

The following proposition characterizes the set of recursive competitive equilibria and the associated equilibrium objects:

Proposition 2. *All recursive competitive equilibria can be indexed by $b_\pi \in [\underline{b}_\pi, \bar{b}_\pi]$ and are characterized as follows. For a given b_π, define the following extended value function $\hat{V} : (-\infty, y/r^\star) \to \mathbb{R}$,*

$$\hat{V}(b) = \begin{cases} \frac{u(y - r^\star b)}{\rho} & \textit{if } b \leq b_\pi \\ \hat{V}(b_\pi) - u'(C_\pi(b_\pi))(b - b_\pi) & \textit{if } b \in (b_\pi, b^*] \\ \frac{u(y - r^\star b) - \psi_0 \bar{\pi}}{\rho} & \textit{if } b \in (b^*, y/\rho), \end{cases}$$

where $b^* = (y - C_\pi(b_\pi))/r^\star$. Define $\bar{b} = \max\{b \leq y/r^\star \,|\underline{V} \leq \hat{v}(b)\}$. Then for any $b_{max} \leq \bar{b}$ and $b_{min} \in \mathbb{R}_-$, define $\overline{\Omega} = (b_{min}, b_{max}]$, and the following constitutes a recursive equilibrium:

[10] To see that $C_\pi(b)$ exists, fix b and consider the function $G(c) = u(y - r^\star b) - u(c) + \psi(\bar{\pi}) + u'(c)(c - y + r^\star b)$, which is the left hand side of (4). Note that $G'(c) > 0$ for $c < y - r^\star b$, $G(y - r^\star b) = \psi(\bar{\pi}) > 0$, and $\lim_{c \downarrow 0} G(c) < 0$ by the condition that $\lim_{c \downarrow 0} u'(c) \to \infty$.

(i) The interest rate schedule $r : \overline{\Omega} \to \{r^\star, r^\star + \bar{\pi}\}$ *defined by*

$$r(b) = \begin{cases} r^\star & \text{if } b \le b_\pi \\ r^\star + \bar{\pi} & \text{if } b \in (b_\pi, b_{max}]; \end{cases}$$

(ii) The value function $V : \overline{\Omega} \to \mathbb{R}$ *defined by* $V(b) = \hat{V}(b)$ *for* $b \in \overline{\Omega}$*;*

(iii) The consumption policy function $C : \overline{\Omega} \to \mathbb{R}_+$ *defined by*

$$C(b) = \begin{cases} y - r^\star b & \text{if } b \le b_\pi \text{ or } b \ge b^\star \\ C_\pi(b_\pi) & \text{if } b \in (b_\pi, b^\star); \text{and} \end{cases}$$

(iv) The inflation policy function $\Pi : \overline{\Omega} \to \{0, \bar{\pi}\}$ *defined by:*

$$\Pi(b) = \begin{cases} 0 & \text{if } b \le b_\pi \\ \bar{\pi} & \text{if } b \in (b_\pi, \bar{b}]. \end{cases}$$

Proposition 2 characterizes the set of possible equilibria, in which each equilibrium is indexed by the value of b_π. That is, each equilibrium corresponds to an interest rate function which has a jump at b_π. If $\underline{b}_\pi \ge \bar{b}$, then inflation is zero for the entire domain $\overline{\Omega}$ as default is preferable to the consequences of inflation. More generally, each value $b_\pi \in [\underline{b}_\pi, \bar{b}_\pi] \cap \overline{\Omega}$ specifies a distinct equilibrium with an interest rate function that jumps up at b_π.

To gain some intuition for the nature of the equilibrium, note that the function $\hat{V}$ in the proposition consists of three segments. For $b \le b_\pi$, $\hat{V}(b)$ is the steady-state value function with low inflation when the government faces a low interest rate. As noted above, low inflation is the best response to $r = r^\star$ for $b \le \bar{b}_\pi$, given this value function. Moreover, this value function and $c = y - r^\star b$ satisfies (HJB) for $b < b_\pi$. Similarly, the final segment of $\hat{V}$ is the steady-state value function with high inflation when the government faces a high interest rate. This function satisfies (HJB) for $b > b_\pi$. These two functions are labelled V_1 and V_3, respectively, in figure 1. While the segments satisfy (HJB) locally, they are not a viscosity solution over the entire domain $\overline{\Omega}$. This is due to the fact that they are not equal b_π, and so stitching V_1 and V_3 together is not continuous. Note that viscosity condition (i) implies that V_1 is an upper bound on the value function in the neighborhood of b_π.

The difference at b_π is equal to the discounted cost of inflation $\frac{\psi_0 \bar{\pi}}{\rho}$. In the neighborhood above b_π, the government's optimal response to the jump in the interest rate is to reduce debt to b_π, and not to remain in the high-inflation zone indefinitely. It therefore will consume less than the steady state consumption level $y-(r(b)-\bar{\pi})b = y-r^\star b$. Given the value function at b_π, we can solve for optimal consumption from (HJB). The consumption that satisfies (HJB) at $\hat{V}(b_\pi)$ is given by $C_\pi(b_\pi)$, introduced in definition 4, which uses the envelope condition $-\hat{V}'(b_\pi^+) = \lim_{b\downarrow b_\pi} u'(C(b)) = u'(C_\pi(b_\pi))$. Note that $\hat{V}'(b_\pi^-) \neq \hat{V}'(b_\pi^+)$, so the value function has a kink at b_π. This kink reflects that consumption equals $y - r^\star b$ to the left of b_π, but is strictly lower to the right given the incentive to save. To ensure that this consumption is indeed the solution to (HJB) at b_π, high inflation must be optimal. This is the case if $\psi_0 < u'(C_\pi(b_\pi))b$ for $b > b_\pi$, which motivates the definition of $\underline{b}_\pi$ in definition 4.

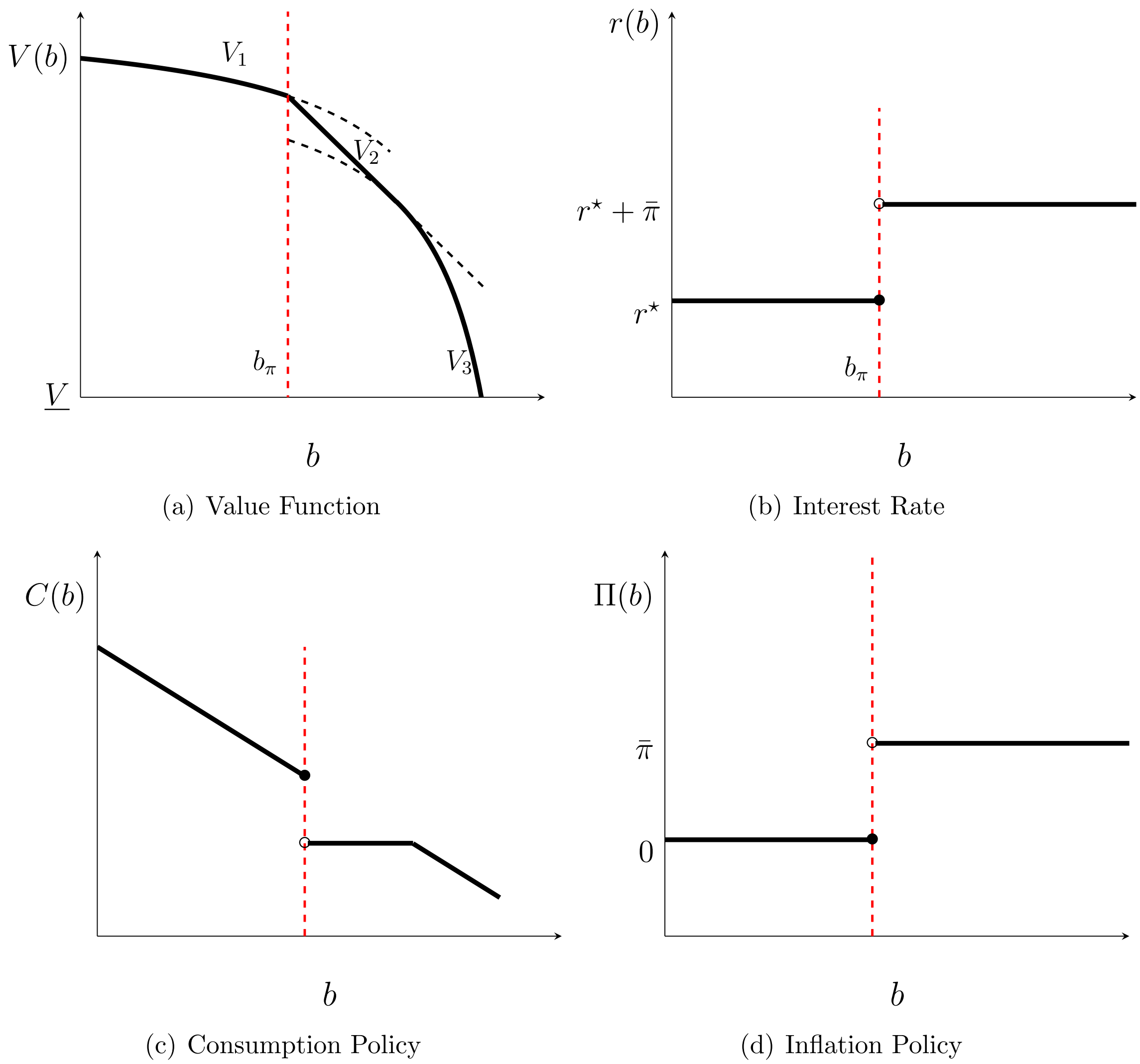
As $r^\star = \rho$, there is no incentive to vary consumption while the government saves. That is, the desire to save is in response to the discontinuity in the interest rate at b_π, not because the current (real) interest rate is high relative to impatience. Thus $C(b) = C_\pi(b_\pi)$ over the domain of active savings, and then jumps to $y-r^\star b_\pi$ at b_π. This is depicted as the horizontal segment of the consumption policy function depicted in figure 1 panel (c). The domain of active savings extends to b^*, at which point $y - C_\pi(b_\pi)b^* = 0$, and consumption is equal to the steady state consumption level. At this level of debt, the government is indifferent to saving towards b_π or remaining at that debt level forever. From the envelope condition, $-\hat{V}'(b) = u'(C_\pi(b_\pi))$ for $b \in (b_\pi, b^*)$; that is, the slope of the value function is constant over this region. This is represented by the linear portion $V_2(b)$ depicted in figure 1. Note that V_2 is tangent to V_3 at b^*, as by definition $C_\pi(b_\pi)$ is the steady state consumption at b^*.

3.1 Comparative Statics

We are interested in how debt dynamics depend on the inflationary regime. Towards this goal, consider an increase in the cost of inflation ψ_0 to $\psi_0' > \psi_0$. To characterize what happens to the set of monotone equilibria, note that the expressions in definition 4 imply that $\underline{b}_\pi$ and $\bar{b}_\pi$ increase. Let $\underline{b}_\pi'$ and $\bar{b}_\pi'$ denote the new thresholds, respectively.

First consider $b_\pi \in [\underline{b}_\pi', \bar{b}_\pi]$; that is, a point of discontinuity that is consistent with equilibrium for both ψ_0 and ψ_0'. The low-inflation steady state value function remain unaffected by the increase in ψ_0, while the high-inflation steady state value function shifts down in a parallel fashion by the amount $\frac{(\psi_0'-\psi_0)\bar{\pi}}{\rho}$. From the expression for C_π in definition 4, $C_\pi(b_\pi)$ declines, which means a higher savings rate and steeper slope associated with the linear portion of the value function. The decline in C_π implies that $b^* = (y - C_\pi(b_\pi))/r^\star$ increases as well, so the domain for the linear portion increases. The steeper slope and larger domain for the linear segment is consistent with the shift down and strict concavity of the high-inflation steady state value function. The new value function is strictly below the original for all $b > b_\pi$. For a given value of $\underline{V}$, this implies that the amount of debt that can be sustained

Figure 1: Government's Solution with No Crisis



(a) Value Function

(b) Interest Rate

(c) Consumption Policy

(d) Inflation Policy

has decreased (as long as $\bar{b}$ is higher than b_π). This is shown in panel (a) of Figure 2.

Consider now what happens to the equilibrium indexed by $b_\pi = \underline{b}_\pi$. The fact that $\underline{b}_\pi < \underline{b}'_\pi$ implies that a discontinuity at $\underline{b}_\pi$ is no longer consistent with equilibrium at the increased cost of inflation ψ'_0. In panel (b) of figure 2 we contrast the value function for an initial equilibrium $b_\pi = \underline{b}_\pi$ with a new equilibrium $b_\pi = \underline{b}'_\pi$. The region $(\underline{b}_\pi, \underline{b}'_\pi]$ shifts from being a high-interest rate zone to a low-interest rate zone. The new optimal policy of low inflation in this zone implies higher welfare, as the government avoids the costs of inflation. That is, the value function is now higher in that region, and by continuity will be higher even at debt levels in which the interest rate jumps up. This reflects the increased proximity to the low-inflation zone. However, given that the linear portion of the value function has a steeper slope under ψ'_0, eventually the new value function intersects the original one from above (see panel (b) of figure 2). Note that depending on the level of $\underline{V}$, the borrowing limit $\bar{b}$ can shift up or down. A similar analysis holds for $b_\pi \in (\bar{b}_\pi, \bar{b}'_\pi]$. Discontinuities at $\bar{b}'_\pi \geq b_\pi > \bar{b}_\pi$ were not consistent with equilibrium under the original ψ, but now become supportable under ψ'_0. This opens a larger potential domain for low interest rates.

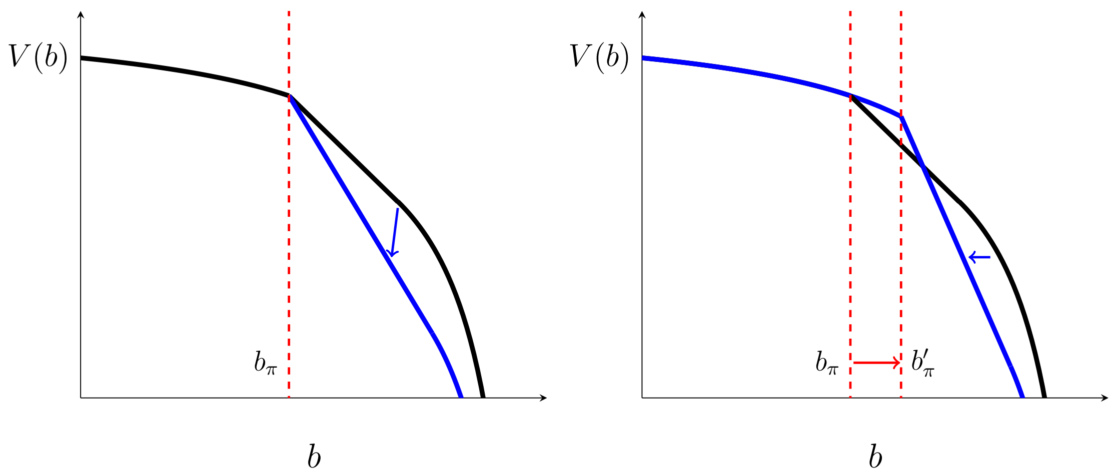
The implication for savings of an increase in ψ is therefore mixed. In panel (a), the savings rate is always weakly greater when ψ_0 is higher, and strictly so for the range $(b_\pi, b^{*\prime})$. In panel (b), when b_π shifts up as a result of the increase in ψ_0, there is a region (b_π, b'_π) in which the low-ψ economy is saving while the high-ψ economy is not. This reflects that inflation rate is higher in this region for the low-ψ economy, and savings is the method to regain commitment to a low inflation rate. As we let ψ_0 go to infinity, the low inflation zone covers the entire space, and savings is zero everywhere. In this limiting case, a strong commitment to low inflation is consistent with weakly higher steady state debt levels and a higher maximal debt limit.

4 Equilibria with Rollover Crises

The preceding analysis constructed equilibria in which bonds were risk free. We now consider equilibria in which investors refuse to purchase new bonds and the government defaults in equilibrium. This links the preceding analysis of nominal bonds with Cole and Kehoe (2000)'s real-bond analysis of self-fulfilling crises and allows us to explore the role of inflation credibility in the vulnerability to debt crises.

Recall that bonds mature at every instant. If investors refuse to roll over outstanding bonds, the government will be unable to repay the debt immediately. However, the government has the option to repay within the grace period δ to avoid the full punishment of default. We first characterize the sub-problem of a government that enters the default state

Figure 2: The Role of Inflation Commitment Absent a Crisis: Increase in ψ



(a) No Change in b_π (b) Change in b_π

but repays the debt within the grace period. We then characterize the government's full problem and characterize equilibria with rollover crises.

4.1 The Grace-Period Problem

To set notation, let $W(b_0, r_0)$ denote the government's value at the start of the grace period with outstanding real bonds b_0 carrying a nominal interest rate r_0. We re-normalize time to zero at the start of the grace period for convenience. To avoid permanent default, the government is obligated to repay the nominal balance on or before date δ, with interest accruing over the grace period at the original contracted rate r_0. We impose the *pari passu* condition that all bond holders have equal standing; that is, the government cannot default on a subset of bonds, while repaying the remaining bondholders. Therefore, the relevant state variable is the entire stock of outstanding debt at the time the government enters the grace period.

The function $W(b_0, r_0)$ is the solution to the following problem:

$$W(b_0, r_0) = \max_{\mathbf{x}\in\mathbf{X}} \int_0^\delta e^{-\rho t} v(x(t))dt + e^{-\rho\delta} V(0), \tag{5}$$

$$\begin{aligned} &\text{subject to :} \\ &\dot{b}(t) = c(t) + (r_0 - \pi(t))b(t) - y \\ &b(0) = b_0, b(\delta) = 0, \dot{b}(t) \leq -\pi(t)b(t). \end{aligned} \tag{6}$$

where for the grace-period problem the controls $\mathbf{x}$ and admissible set $\mathbf{X}$ refer to measurable functions $[0, \delta] \to X$. The $V(0)$ in the objective function represents the equilibrium value of returning to the markets with zero debt (which is to be determined below in equilibrium) at the end of the grace period. Note that if the government repays before the end of the grace period, it could exit default sooner. However, as it has no incentive to borrow again once $b = 0$, it is not restrictive to impose no new debt for the entire grace period. The final constraint, $\dot{b}(t) \leq -\pi(t)b(t)$ is equivalent to the constraint of no new nominal bonds, $\dot{B}(t) \leq 0$.

The grace period problem is a simple finite-horizon optimization with a terminal condition for the state variable. We do not discuss the solution in depth, but highlight a few key implications. An important feature of (5) is that $W(b_0, r_0)$ is strictly decreasing in both arguments. Moreover, for a given value of $V(0)$, W is decreasing in ψ_0, and strictly decreasing if positive inflation is chosen for a non-negligible fraction of the grace period. In order to repay its debt quickly, the government has an incentive to inflate away a portion of the outstanding debt. The cost of doing so is governed by ψ_0.

Regarding a piece of unfinished business left over from the no-crisis analysis, with W in hand we can state explicitly why the government would never choose to enter default in the non-crisis equilibria discussed in the previous section. In particular, the government could always mimic the grace period policy in equilibrium. The one caveat is that r_0 is held constant in the grace period, while the equilibrium interest rate varies with b outside of default. However, as debt is strictly decreasing and $r(b)$ must be monotone in any no-crisis equilibrium, this caveat works against choosing to default.

4.2 Rollover Crises

If investors do not roll over outstanding bonds, the government will be forced to default, but may decide to repay within the grace period to avoid $\underline{V}$. If such an event occurs at time t, then the government will repay within the grace period if and only if $W(b_t, r_t) \geq \underline{V}$. The weak inequality assumes that the government repays if indifferent.

We assume that a rollover crisis is an equilibrium possibility only if $W(b_t, r_t) < \underline{V}$. This equilibrium selection assumption is motivated as follows. Suppose that lenders call in their bonds and the government repays within the grace period. The outstanding bonds would carry a positive price in a secondary market and individual lenders would be willing to purchase new bonds at the margin from the government at a positive price. Indeed, the face value of the bonds will be paid in full. It is an artifact of continuous time that a rollover crisis induces (temporary) default in a region of the state space in which the government is willing to honor all nominal obligations within the specified interval of time. Such crises would not survive in discrete-time equilibria. In particular, a grace period of $\delta = 1$ in a discrete time formulation with one-period bonds would rule out all such crises (as in Cole and Kehoe (2000)). We avoid such artificial outcomes through the equilibrium selection assumption.

On the other hand, a rollover crisis when $W(b_t, r_t) < \underline{V}$ has a natural interpretation. If this inequality holds and all other investors refuse to roll over their bonds, an individual lender would have no incentive to extend new credit to the government. Assuming each lender is infinitesimal, such new loans would not change the government's default decision. Moreover, as the government would not repay this new debt, such lending would not be challenged by outstanding bondholders. Such crises would survive in a discrete-time formulation.

Similar to Cole and Kehoe (2000) we assume that, as long as $W(b_t, r(b_t)) < \underline{V}$, a rollover crisis occurs with a Poisson arrival probability equal to λ. The value of λ will be taken as a primitive in the definition of an equilibrium below, as is δ, the grace period. We can define an indicator function for the region in which outright default is preferable to repayment within the grace period:

Definition 5. *Let $I : \mathbb{R}^2 \to \{0, 1\}$ be defined as follows:*

$$I(b_0, r_0) = \begin{cases} 1 & \textit{if } W(b_0, r_0) < \underline{V} \\ 0 & \textit{otherwise} \end{cases}$$

The Poisson probability of a crisis at time t can then be expressed as $\lambda I(b_t, r_t)$. Given an equilibrium $r(b)$, we shall refer to the set $\{b \in \overline{\Omega} | I(b, r(b)) = 1\}$ as the "crisis zone," and its complement in $\overline{\Omega}$ as the "safe zone."

4.3 The Government's Problem

We now state the problem of the government when not in default. As in the no-crisis equilibrium of section 3, we assume the government faces a bond-market equilibrium characterized by domain $\overline{\Omega}$ and a $r \in \mathcal{R}(\overline{\Omega})$, as well as the parameters δ and λ defining the duration of the grace period and the Poisson probability bonds are called conditional on $I(b_t, r(b_t)) = 1$. Let $T \in (0, \infty]$ denote the first time loans are called (i.e., a rollover crisis occurs). From the government's and an individual creditor's perspective, T is a random variable with a distribution that depends on the path of the state variable. In particular, $\Pr(T \leq \tau) = 1 - e^{-\lambda \int_0^\tau I(b(t), r(b(t)))dt}$. The realization of T is public information and it is the only uncertainty in the model. The government's problem is:

$$
\begin{aligned}
V(b_0) = \max_{\mathbf{x} \in \mathbf{X}} & \left\{ \int_0^\infty e^{-\lambda \int_0^t I(b(s), r(b(s))ds - \rho t} v(x(t))dt \right. \\
& \left. + \lambda \underline{V} \int_0^\infty e^{-\lambda \int_0^t I(b(s), r(b(s)))ds - \rho t} dt \right\} \\
& \text{subject to:} \\
& b(t) = b_0 + \int_0^t f(b(t), x(t))dt \text{ and} \\
& b(t) \in \overline{\Omega} \text{ for all } t.
\end{aligned} \tag{P2}
$$

As in the non-crisis case, we impose the equilibrium restriction on $\overline{\Omega}$ that default is never optimal.[11]

The associated Bellman equation is:

$$
\begin{aligned}
(\rho + \lambda I_b)V(b) - \lambda I_b \underline{V} &= max_{x \in X} \left\{ v(x) - V'(b) f(b, x) \right\} \\
&= \max_{(c,\pi) \in X} \left\{ u(c) - \psi(\pi) - V'(b) \left(c + (r(b) - \pi)b - y \right) \right\},
\end{aligned} \tag{HJB'}
$$

where I_b is shorthand for the crisis indicator $I(b, r(b))$. As in the no-crisis case, the government's value function is the unique solution to this equation:

Proposition 3. *For a given $\overline{\Omega}$ and $r \in \mathcal{R}(\overline{\Omega})$, the government's value function defined in (P2) is the unique bounded Lipschitz-continuous viscosity solution to (HJB').*

4.4 Crisis Equilibrium

We can now state the definition of equilibrium with crisis:

[11] That is, $V(b) \geq \max\langle \underline{V}, W(b, r(b)) \rangle$ for all $b \in \overline{\Omega}$ will be satisfied in any equilibrium.

Definition 6. *A* Recursive Competitive Equilibrium with Crisis *is an interval* $\overline{\Omega} = [b_{min}, b_{max}]$, *an interest rate schedule* r, *a consumption policy function* $C : \overline{\Omega} \to [0, \bar{c}]$, *an inflation policy function* $\Pi : \overline{\Omega} \to [0, \bar{\pi}]$, *and a value function* $V : \overline{\Omega} \to \mathbb{R}$ *such that*

(i) $r \in \mathcal{R}(\overline{\Omega})$;

(ii) given $(\overline{\Omega}, r)$ *and for any* $b_0 \in \overline{\Omega}$, *the policy functions combined with the law of motion* (1) *and initial debt* b_0 *generate sequences* $x(t) = (C(b(t)), \Pi(b(t)))$ *that solve the government's problem* (P2) *and deliver* $V(b_0)$ *as a value function;*

(iii) given $C(b)$ *and* $\Pi(b)$, *bond holders earn a real return* $r^\star$, *that is,* $r(b) = r^\star + \Pi(b) + \lambda I(b, r(b))$ *for all* $b \in \overline{\Omega}$; *and*

(iv) $V(b_0) \geq \underline{V}$ *for all* $b \in \overline{\Omega}$.

Note that when $\lambda = 0$ this equilibrium corresponds to the equilibrium in Definition 2.

As in section 3, we can restrict attention to equilibria in which $r(b)$ takes discrete values:

Lemma 2. *In any equilibrium with crisis,* $r(b) \in \{r^\star, r^\star + \bar{\pi}, r^\star + \lambda, r^\star + \bar{\pi} + \lambda\}$ *for all* $b \in \overline{\Omega}$.

In contrast to the non-crisis case in the previous section, we can construct non-monotone equilibria. In particular, $r(b)$ need not be monotonic in the crisis zone.We restrict attention to monotone equilibria; that is, equilibria in which $r(b)$ is non-decreasing. As W is strictly decreasing in both arguments, monotonicity in $r(b)$ ensures that $I(b, r(b))$ is non-decreasing as well, and the safe zone can be defined as an interval $[b_{min}, b_\lambda]$ for some $b_\lambda \in \mathbb{R}_{++}$. This threshold for the safe zone can be characterized as follows. Define $\underline{b}_\lambda$ and $\bar{b}_\lambda$ by:

Definition 7. *Let*

$$\underline{b}_\lambda \equiv \sup \left\{ b \leq \frac{(1 - e^{-r^\star \delta})y}{\rho} \middle| W(b, r^\star + \bar{\pi}) \geq \underline{V} \right\}; \textit{and}$$
$$\bar{b}_\lambda \equiv \sup \left\{ b \leq \frac{(1 - e^{-r^\star \delta})y}{\rho e^{-\bar{\pi}\delta}} \middle| W(b, r^\star) \geq \underline{V} \right\}.$$

These two thresholds correspond to the maximal debt the government is willing to repay within the grace period if the interest rate is $r^\star + \bar{\pi}$ and $r^\star$, respectively. From the government's problem described in section 4.1, we have $\underline{b}_\lambda < \bar{b}_\lambda$. This follows from the fact that W is strictly decreasing in both arguments. As we shall see, the equilibrium threshold for a rollover crisis $b_\lambda \in [\underline{b}_\lambda, \bar{b}_\lambda]$, the exact value within this interval being determined by optimal inflation.

We now turn to two thresholds that determine the optimal inflation policy. We know from the analysis of section 3 that there is an indeterminacy regarding the threshold for inflation,

b_π. We consider equilibria in which the low inflation zone is as large as possible. In the non-crisis equilibria, the maximum threshold is $\bar{b}_\pi$ from definition 4, which is the maximal debt consistent with zero inflation when the government is offered an interest rate of $r^\star$. With the possibility of a rollover crisis, we introduce a second threshold, $\tilde{b}_\pi$. This threshold concerns the best response when the interest rate is $r^\star + \lambda$. As we shall see, and consistent with the no-crisis analysis, an increase in the interest rate from $r^\star$ to $r^\star + \lambda$ provides an incentive to save and a corresponding incentive to inflate. We define the inflation threshold $\tilde{b}_\pi$ to be consistent with a discontinuity in r at $\bar{b}_\lambda$; we shall see that this is the relevant threshold in equilibrium. Specifically,

Definition 8. *Let $\tilde{b}_\pi$ be defined as:*

$$\tilde{b}_\pi = \begin{cases} \frac{\psi_0}{u'(c_\lambda)} & \text{if } c_\lambda \le y - \frac{(r^\star+\lambda)\psi_0}{u'(c_\lambda)} \\ \frac{\psi_0}{u'(y-(r^\star+\lambda)\tilde{b}_\pi)} & \text{otherwise,} \end{cases}$$

where $c_\lambda \in (0, y - (r^\star + \lambda)\bar{b}_\lambda)$ is defined uniquely by

$$\frac{(\rho+\lambda)u(y - r^\star\bar{b}_\lambda)}{\rho} = u(c_\lambda) - u'(c_\lambda)(c_\lambda - y + r^\star\bar{b}_\lambda) + \lambda\underline{V}.$$

The consumption c_λ satisfies (HJB') as we approach $\bar{b}_\lambda$ from above if $V(\bar{b}_\lambda) = u(y - r^\star\bar{b}_\lambda)/\rho$, $r(b) = r^\star + \lambda$ and $\pi = 0$. At this consumption level, $\pi = 0$ is optimal if $\psi_0 \ge u'(c_\lambda)b$. If $b > (y - c_\lambda)/(r^\star + \lambda)$, then the government would prefer the steady-state consumption $y - (r^\star + \lambda)b$ to c_λ, hence the second line in the expression for $\tilde{b}_\pi$. The threshold $\tilde{b}_\pi$ is the maximum debt when there is the possibility of a crisis and yet the government opts for low inflation. Note that $\tilde{b}_\pi < \bar{b}_\pi$, as $c_\lambda < y - (r^\star + \lambda)\bar{b}_\lambda < y - r^\star\bar{b}_\lambda$, where the last term is the steady-state consumption when $r(b) = r^\star$.

The equilibrium will depend on the relative magnitudes of these four thresholds. In particular, the thresholds define the maximum debt levels consistent with no crisis and/or low inflation, and so will be useful in defining the greatest domain for low interest rates.[12]Before characterizing the equilibria in full, we discuss their general properties regarding inflation and vulnerability to rollover crises.

Any monotone equilibrium $r(b)$ will be characterized by $\{b_\pi, b_\lambda\}$ that determine the edge of the low-inflation and safe zones, respectively. From the above discussion, $b_\pi \in [\tilde{b}_\pi, \bar{b}_\pi]$ and $b_\lambda \in [\underline{b}_\lambda, \bar{b}_\lambda]$. Figure 3 presents the four possible configurations for the thresholds. Each panel

[12] In section 3 we also considered $\underline{b}_\pi$, the smallest domain for a low interest rate. As creditors are indifferent and the government prefers a low interest rate, the maximal domain is weakly Pareto superior. We focus in this section on these upper-bound thresholds, tracing out the Pareto-dominant equilibrium interest rate function, conditional on parameters.

depicts the crisis cutoffs $\{\underline{b}_\lambda, \bar{b}_\lambda\}$ and the inflation cutoffs $\{\tilde{b}_\pi, \bar{b}_\pi\}$. While we know $\underline{b}_\lambda < \bar{b}_\lambda$ and $\tilde{b}_\pi < \bar{b}_\pi$, the magnitude of the inflation thresholds relative to the crisis thresholds depends on parameters. The four panels of figure 3 depict four possible cases.

Case 1 is depicted in panel (a). In this case, ψ_0 is low and so $b_\pi = \bar{b}_\pi < \underline{b}_\lambda$. In case 1, inflation is high for part of the safe zone. The relevant crisis threshold is therefore $b_\lambda = \underline{b}_\lambda$. That is, the crisis threshold is determined by $W(b, r^\star + \bar{\pi})$, as inflation is high at the relevant debt level.

Recall that the inflation cutoffs are strictly increasing in ψ_0 and that the crisis thresholds are weakly decreasing (and strictly decreasing if inflation is optimal in the grace-period problem). As ψ_0 increase, the inflation thresholds shift right, and the crisis thresholds (at least weakly) shift left, which corresponds to the movement from panel (a) to panel (b) in the figure. Once $\underline{b}_\lambda < \bar{b}_\pi$, $r(b) = r^\star$ at $\underline{b}_\lambda$. Therefore, the threshold $\underline{b}_\lambda$ is not relevant. However, inflation jumps at $b_\pi = \bar{b}_\pi$. If $\underline{b}_\lambda < \bar{b}_\pi$, then the associated jump in the interest rate is sufficient to generate crises. In panel (b), we depict the case $\underline{b}_\lambda < \bar{b}_\pi < \bar{b}_\lambda$, and so this jump at $\bar{b}_\pi = b_\pi = b_\lambda$ defines the crisis zone. This is case (2).

For higher ψ_0, $\bar{b}_\pi > \bar{b}_\lambda$. At $\bar{b}_\lambda$, a crisis becomes possible even if $\pi = 0$. In panel (c), $b_\lambda = \bar{b}_\lambda$ defines the crisis zone. Moreover, the fact that $\bar{b}_\lambda > \tilde{b}_\pi$ implies that the optimal response to being in the crisis zone involves inflation. Therefore $\bar{b}_\lambda = b_\pi$ also defines the inflation zone (case 3). However, if ψ_0 is large enough, $\tilde{b}_\pi > \bar{b}_\lambda$, and then inflation is high only for $b > \tilde{b}_\pi = b_\pi$, which is case (4) in panel (d).

The four thresholds as functions of the parameter ψ_0 are depicted in figure 4. The portions in bold refer to the equilibrium threshold for crisis $\underline{b}$ (panel (a)) and inflation b_π (panel (b)). As noted above, the crisis thresholds are (weakly) decreasing in ψ_0 with $\underline{b}_\lambda < \bar{b}_\lambda$, and the inflation thresholds are strictlty increasing in ψ_0, with $\tilde{b}_\pi < \bar{b}_\pi$. There are three values of ψ_0 that are of interest:

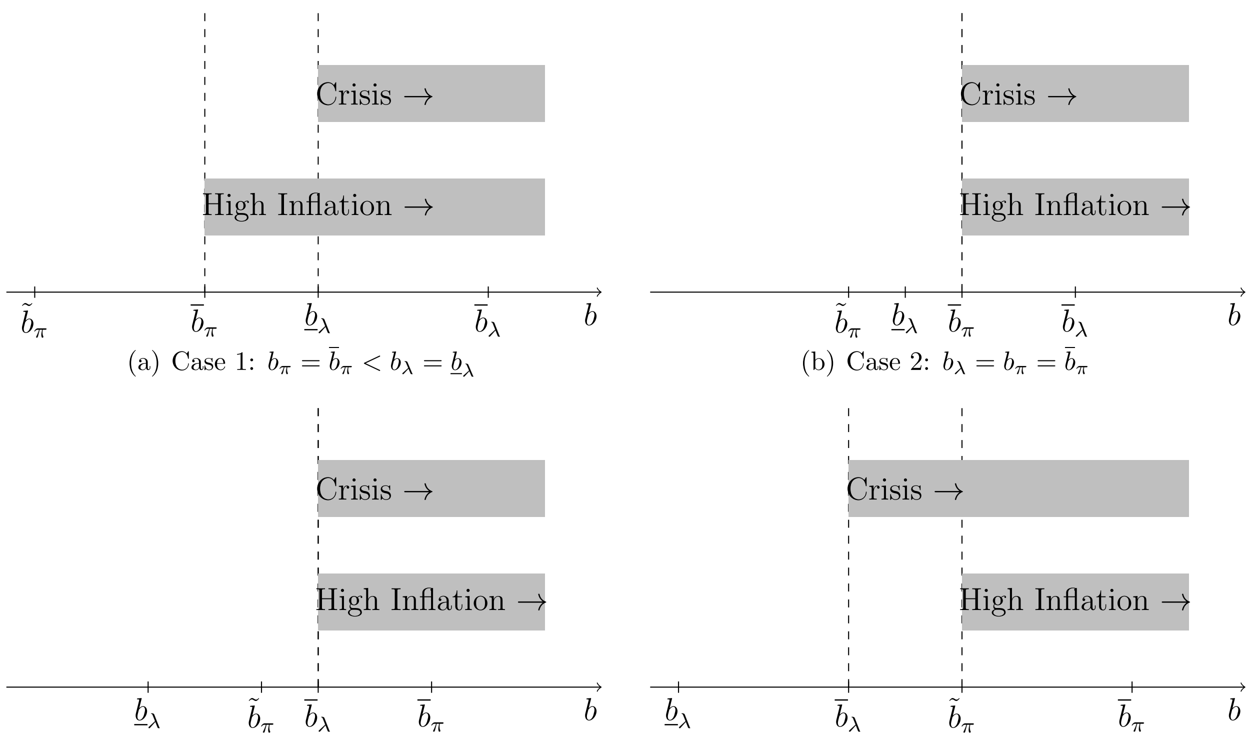
Definition 9. *Define $\boldsymbol{\psi_1}$ as the cost of inflation such that $\bar{b}_\pi = \underline{b}_\lambda$; define $\boldsymbol{\psi_2}$ as the cost of inflation such that $\bar{b}_\pi = \bar{b}_\lambda$; and define $\boldsymbol{\psi_3}$ as the cost of inflation such that $\tilde{b}_\pi = \bar{b}_\lambda$.*

Note that $\psi_1 < \psi_2 < \psi_3$. These three values divide the parameter space into four regions. The next four propositions characterize the equilibria in the four possible cases. As the propositions share many similarities, we redefine notation when convenient. After each proposition, we discuss the characteristics of the equilibrium before moving to the next case.

Case 1: $\psi_0 \in [0, \psi_1]$

We now characterize equilibria for $\psi_0 < \psi_1$, which relates to panel (a) of figure 3:

Figure 3: Thresholds for Inflation and Crisis



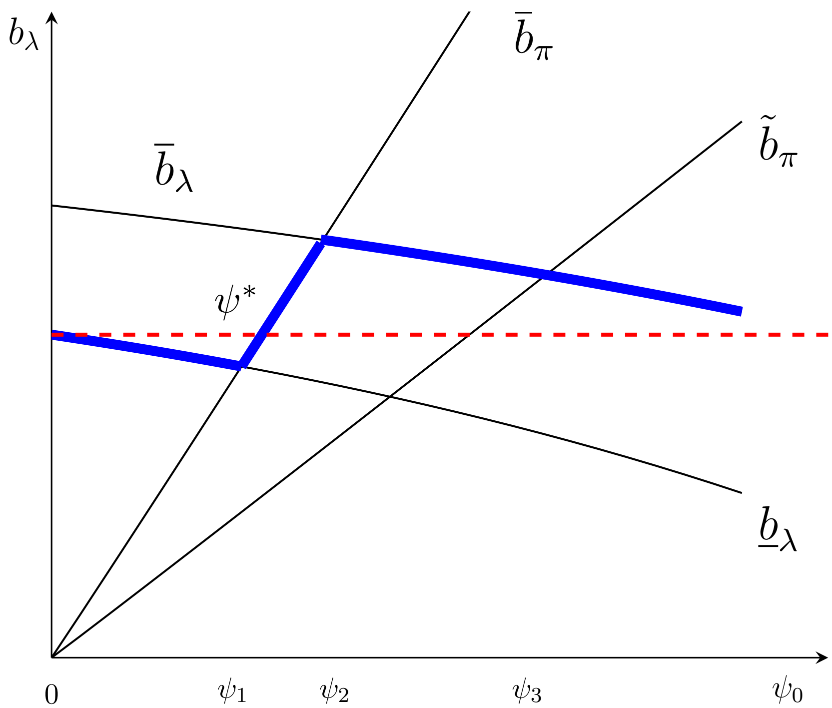
(a) Case 1: $b_\pi = \bar{b}_\pi < b_\lambda = \underline{b}_\lambda$

(b) Case 2: $b_\lambda = b_\pi = \bar{b}_\pi$

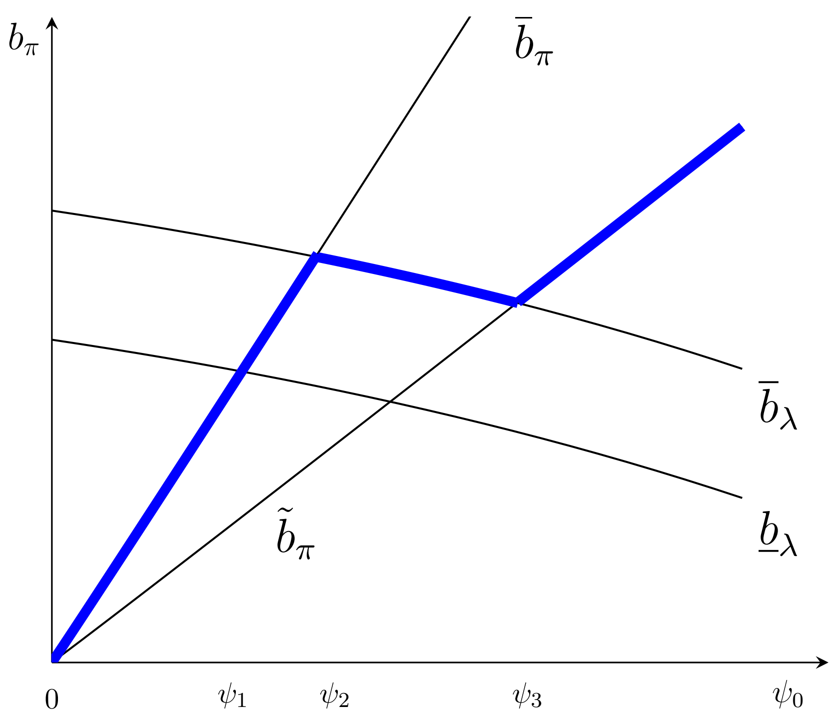
(c) Case 3: $b_\lambda = b_\pi = \bar{b}_\lambda$

(d) Case 4: $b_\lambda = \bar{b}_\lambda < b_\pi = \tilde{b}_\pi$

Figure 4: Thresholds as a Function of Inflation Commitment



(a) $\underline{b}$ as a Function of ψ_0



(b) b_π as a Function of ψ_0

Proposition 4. *Suppose $\bar{b}_\pi \leq \underline{b}_\lambda$ (that is, $\psi_0 \in [0, \psi_1]$). Define $c_\pi = C_\pi(\bar{b}_\pi)$, where $C_\pi(b)$ is as in definition 4. Define $b^*_\pi = (y - c_\pi)/r^\star$. For $b \leq \underline{b}_\lambda$, define $\hat{V}(b)$ by*[13]

$$\hat{V}(b) = \begin{cases} \frac{u(y-r^\star b)}{\rho} & \textit{if } b \leq \bar{b}_\pi \\ \frac{u(y-r^\star \bar{b}_\pi)}{\rho}) - u'(c_\pi))(b - \bar{b}_\pi) & \textit{if } b \in (\bar{b}_\pi, \min\langle b^*, \underline{b}_\lambda\rangle) \\ \frac{u(y-r^\star b)-\psi_0\bar{\pi}}{\rho} & \textit{if } b \in [b^*, \underline{b}_\lambda], \end{cases}$$

Define $c_\lambda \in (0, y - (r^\star + \lambda)\underline{b}_\lambda)$ as the solution to

$$(\rho + \lambda)\hat{V}(\underline{b}_\lambda) = u(c_\lambda) - \psi_0\bar{\pi} - u'(c_\lambda)(c_\lambda + (r^\star + \lambda)\underline{b}_\lambda - y) + \lambda\underline{V}.$$

Let $b^*_\lambda = (y - c_\lambda)/(r^\star + \lambda)$. For $b > \underline{b}_\lambda$, define $\hat{V}(b)$ by

$$\hat{V}(b) = \begin{cases} V(\underline{b}_\lambda) - u'(c_\lambda)(b - \underline{b}_\lambda) & \textit{if } b \in (\underline{b}_\lambda, b^*_\lambda) \\ \frac{u(y-(r^\star+\lambda)b)-\psi_0\bar{\pi}}{\rho+\lambda} + \frac{\lambda}{\rho+\lambda}\underline{V} & \textit{if } b \geq b^*_\lambda. \end{cases}$$

Define $b_{max} = \max\{b \leq y/(r^\star + \lambda) | \underline{V} \leq \hat{V}(b)\}$. Then define $\overline{\Omega} = [b_{min}, b_{max}]$ for $b_{min} \in \mathbb{R}_-$, and the following constitutes a recursive equilibrium with crisis parameter λ:

(i) The interest rate schedule $r : \overline{\Omega} \to \{r^\star, r^\star + \bar{\pi}, r^\star + \bar{\pi} + \lambda\}$ defined by

$$r(b) = \begin{cases} r^\star & \textit{if } b \in [b_{min}, \bar{b}_\pi] \cap \overline{\Omega} \\ r^\star + \bar{\pi} & \textit{if } b \in (\bar{b}_\pi, \underline{b}_\lambda] \cap \overline{\Omega} \\ r^\star + \bar{\pi} + \lambda & \textit{if } b \in (\underline{b}_\lambda, b_{max}] \cap \overline{\Omega}; \end{cases}$$

(ii) The value function $V : \overline{\Omega} \to \mathbb{R}$ defined by $V(b) = \hat{V}(b)$ for $b \in \overline{\Omega}$;

(iii) The consumption policy function $C : \overline{\Omega} \to \mathbb{R}_+$ defined by

$$C(b) = \begin{cases} y - r^\star b & \textit{if } b \in [b_{min}, \bar{b}_\pi] \cap \overline{\Omega} \\ c_\pi & \textit{if } b \in (b_\pi, \min\langle b^*, \underline{b}_\lambda\rangle] \cap \overline{\Omega} \\ y - r^\star b & \textit{if } b \in (b^*, \underline{b}_\lambda] \cap \overline{\Omega} \\ c_\lambda & \textit{if } b \in (\underline{b}_\lambda, b^*_\lambda] \cap \overline{\Omega} \\ y - (r^\star + \lambda)b & \textit{if } b \in (b^*_\lambda, b_{max}] \cap \overline{\Omega}; \end{cases}$$

[13] In defining $\hat{V}$ in each proposition, for notational ease we do not include the restrictions on debt that ensure consumption is non-negative. As we later truncate $\hat{V}$ to a domain on which consumption is positive, this extended domain is not relevant to the equilibrium characterization.

(iv) The inflation policy function $\Pi : \overline{\Omega} \to \{0, \bar{\pi}\}$ *defined by:*

$$\Pi(b) = \begin{cases} 0 & \text{if } \in b \in [b_{min}, \bar{b}_\pi] \cap \overline{\Omega} \\ \bar{\pi} & \text{if } b \in (\bar{b}_\pi, b_{max}] \cap \overline{\Omega}. \end{cases}$$

The equilibrium is depicted in figure 5. In the case of $\bar{b}_\pi < \underline{b}_\lambda$, the government has an incentive to inflate in a region in which there is no probability of a crisis, reflecting the low level of inflationary commitment. This implies that in the region $b \le \underline{b}_\lambda$, the analysis is the same as in section 3. For low debt, the government does not inflate and enjoys steady-state utility. This is the first segment of the value function depicted in figure 5. Low inflation is no longer optimal for $b > \bar{b}_\pi$, and inflation and the interest rate respond accordingly. As in the no-crisis case of section 3, this jump in inflation and the corresponding increase in the interest rate provides an incentive to save. In the neighborhood above $\bar{b}_\pi$, consumption is constant at c_π as the economy saves towards this threshold, with consumption satisfying the corresponding Bellman equation. If the distance between $\bar{b}_\pi$ and $\underline{b}_\lambda$ is large enough (which is not the case depicted in figure 5), there may be a high-inflation/no-crisis region where the government sets $\dot{b} = 0$ (i.e., $(b^*, \underline{b}_\lambda]$). Given the high debt levels and the low consumption, the government's optimal policy is to inflate, rationalizing the jump in the interest rate as an equilibrium.

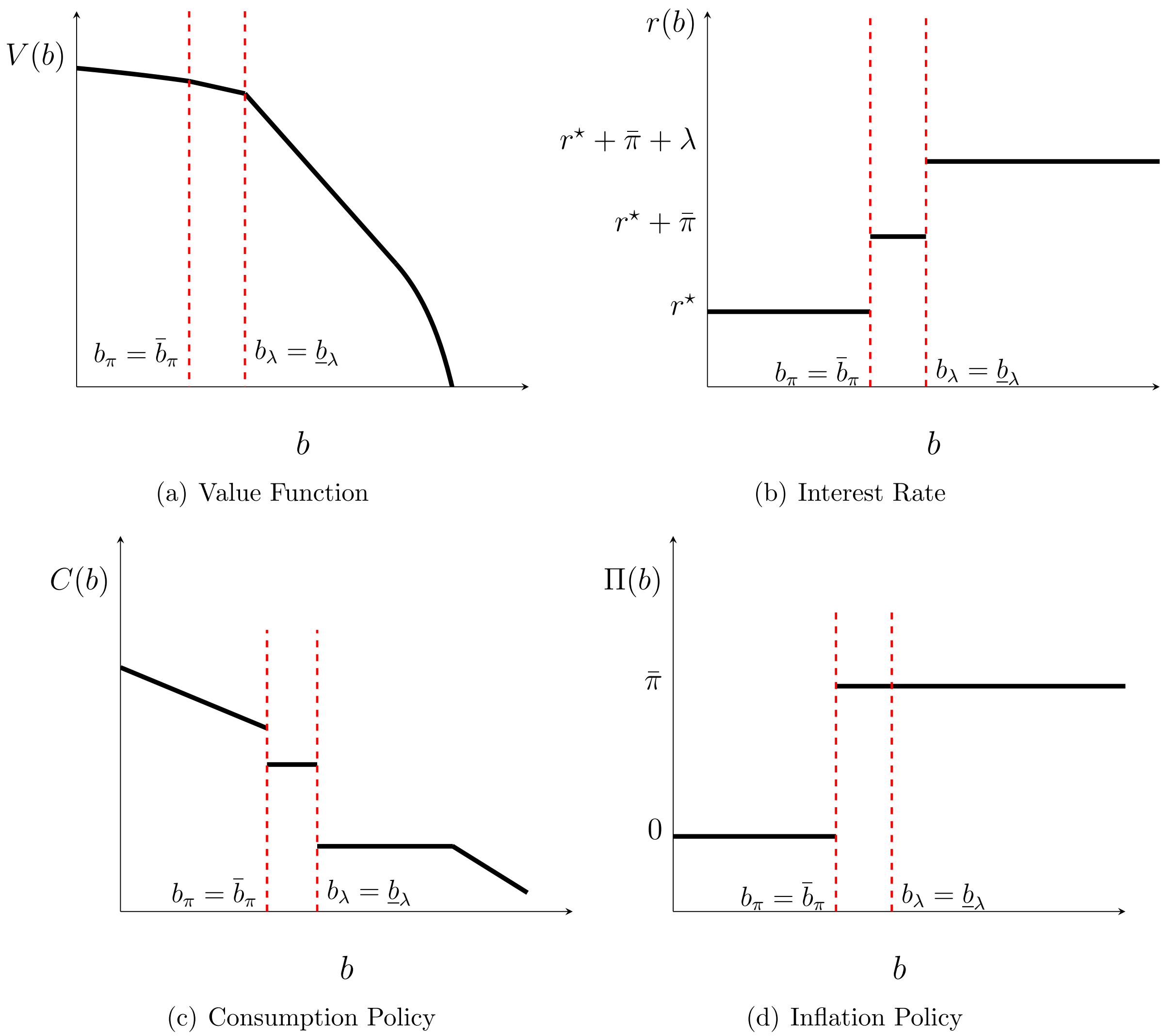
At debt greater than $\underline{b}_\lambda$, the economy is vulnerable to a rollover crisis. The interest rate jumps again to $r^\star + \bar{\pi} + \lambda$. This provides the government with a greater incentive to save, and reflects the kink at b_λ, after which the value function declines more rapidly. The corresponding consumption level is $c_\lambda < c_\pi$, which satisfies the Bellman equation at $\underline{b}_\lambda$. Note that consumption is discretely lower at $\underline{b}_\lambda$, so inflation is weakly greater, verifying that $\bar{\pi}$ is optimal in the crisis zone as well. The equilibrium behavior of the government therefore is to save in a neighborhood above $\underline{b}_\lambda$ to eliminate the possibility of a crisis as well as reduce inflation; at $\underline{b}_\lambda$, it may continue to save at a slower rate in order to reduce inflation, eventually reaching $\bar{b}_\pi$.

Case 2: $\psi_0 \in (\psi_1, \psi_2]$

Proposition 5. *Suppose* $\bar{b}_\pi \in (\underline{b}_\lambda, \bar{b}_\lambda]$ *(that is,* $\psi_0 \in (\psi_1, \psi_2]$*). Define* $c_\pi \in (0, y - r^\star \bar{b}_\pi)$ *as the solution to*

$$\frac{(\rho + \lambda) u(y - r^\star \bar{b}_\pi)}{\rho} = u(c_\pi) - \psi_0 \bar{\pi} - u'(c_\pi)(c_\pi + (r^\star + \lambda)\bar{b}_\pi - y) + \lambda \underline{V}.$$

Figure 5: Case 1: Crisis Equilibrium if $\psi_0 \in [0, \psi_1]$



(a) Value Function (b) Interest Rate (c) Consumption Policy (d) Inflation Policy

Let $b^* = (y - c_\pi)/(r^\star + \lambda)$. Define $\hat{V}(b)$ by

$$\hat{V}(b) = \begin{cases} \frac{u(y - r^\star b)}{\rho} & \text{if } b \leq \bar{b}_\pi \\ \frac{u(y - r^\star \bar{b}_\pi)}{\rho} - u'(c_\pi)(b - \bar{b}_\pi) & \text{if } b \in (\bar{b}_\pi, b^*) \\ \frac{u(y - (r^\star + \lambda)b) - \psi_0 \bar{\pi}}{\rho + \lambda} + \frac{\lambda}{\rho + \lambda}\underline{V} & \text{if } b \geq b^*. \end{cases}$$

Define $b_{max} = \max\{b \leq y/(r^\star + \lambda) | \underline{V} \leq \hat{V}(b)\}$. Then define $\overline{\Omega} = [b_{min}, b_{max}]$ for $b_{min} \in \mathbb{R}_-$, and the following constitutes a Recursive Equilibrium with Crisis:

(i) The interest rate schedule $r : \overline{\Omega} \to \{r^\star, r^\star + \bar{\pi} + \lambda\}$ defined by

$$r(b) = \begin{cases} r^\star & \text{if } b \in [b_{min}, \bar{b}_\pi] \cap \overline{\Omega} \\ r^\star + \bar{\pi} + \lambda & \text{if } b \in (\bar{b}_\pi, b_{max}] \cap \overline{\Omega}; \end{cases}$$

(ii) The value function $V : \overline{\Omega} \to \mathbb{R}$ defined by $V(b) = \hat{V}(b)$ for $b \in \overline{\Omega}$;

(iii) The consumption policy function $C : \overline{\Omega} \to \mathbb{R}_+$ defined by

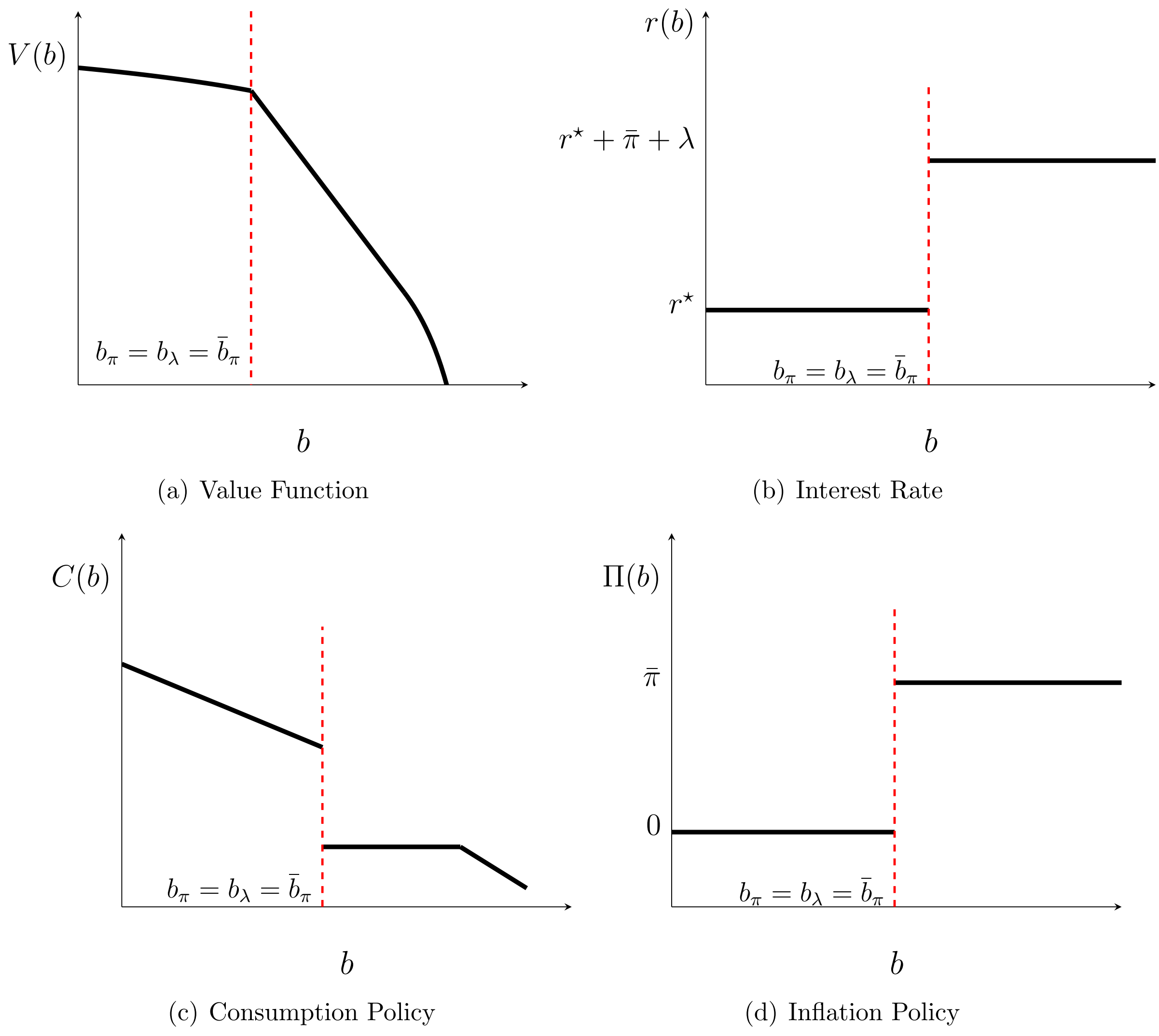
$$C(b) = \begin{cases} y - r^\star b & \text{if } b \in [b_{min}, \bar{b}_\pi] \cap \overline{\Omega} \\ c_\pi & \text{if } b \in (\bar{b}_\pi, b^*] \cap \overline{\Omega} \\ y - (r^\star + \lambda)b & \text{if } b \in (b^*, b_{max}] \cap \overline{\Omega}; \end{cases}$$

(iv) The inflation policy function $\Pi : \overline{\Omega} \to \{0, \bar{\pi}\}$ defined by:

$$\Pi(b) = \begin{cases} 0 & \text{if } b \in [b_{min}, \bar{b}_\pi] \cap \overline{\Omega} \\ \bar{\pi} & \text{if } b \in (b_\pi, b_{max}] \cap \overline{\Omega}. \end{cases}$$

In this case, the economy has low inflation at $\underline{b}_\lambda$, so this is not the relevant threshold for the safe zone. However, inflation may be high in equilibrium at $\bar{b}_\lambda$, making this an irrelevant threshold as well. We have instead that the equilibrium threshold for a crisis is $b_\lambda = \bar{b}_\pi$, so the jump in the interest rate due to high inflation creates room for a crisis. The government's value function is depicted in figure 6. The government is at a low inflation steady state for $b \leq \bar{b}_\pi = b_\lambda$. At $b \in (b_\lambda, b_\lambda + \varepsilon)$ for some $\varepsilon > 0$ the economy saves towards the low inflation/safe zone, setting $\pi = \bar{\pi}$. Consumption is c_π with $\pi = \bar{\pi}$ and $V(b_\lambda) = \frac{u(y - r^\star b_\lambda)}{\rho}$.

Figure 6: Case 2: Crisis Equilibrium if $\psi_0 \in (\psi_1, \psi_2]$



Case 3: $\psi_0 \in (\psi_2, \psi_3]$

Proposition 6. *Suppose $\bar{b}_\pi > \bar{b}_\lambda \geq \tilde{b}_\pi$ (that is, $\psi_0 \in (\psi_2, \psi_3]$). Define $c_\lambda \in (0, y - r^\star \bar{b}_\lambda)$ as the solution to*

$$\frac{(\rho+\lambda)u(y - r^\star \bar{b}_\lambda)}{\rho} = u(c_\lambda) - \psi_0\bar{\pi} - u'(c_\lambda)(c_\lambda + (r^\star + \lambda)\bar{b}_\lambda - y) + \lambda \underline{V}.$$

Let $b^* = (y - c_\lambda)/(r^\star + \lambda)$. Define $\hat{V}(b)$ by

$$\hat{V}(b) = \begin{cases} \frac{u(y - r^\star b)}{\rho} & \text{if } b \leq \bar{b}_\lambda \\ \frac{u(y - r^\star \bar{b}_\lambda)}{\rho} - u'(c_\lambda)(b - \bar{b}_\lambda) & \text{if } b \in (\bar{b}_\lambda, b^*) \\ \frac{u(y - (r^\star + \lambda)b) - \psi_0\bar{\pi}}{\rho + \lambda} + \frac{\lambda}{\rho+\lambda}\underline{V} & \text{if } b \geq b^*. \end{cases}$$

Define $b_{max} = \max\{b \leq y/(r^\star + \lambda) | \underline{V} \leq \hat{V}(b)\}$. Then define $\overline{\Omega} = [b_{min}, b_{max}]$ for $b_{min} \in \mathbb{R}_-$, and the following constitutes a Recursive Equilibrium with Crisis:

(i) The interest rate schedule $r : \overline{\Omega} \to \{r^\star, r^\star + \bar{\pi} + \lambda\}$ defined by

$$r(b) = \begin{cases} r^\star & \text{if } b \in [b_{min}, \bar{b}_\lambda] \cap \overline{\Omega} \\ r^\star + \bar{\pi} + \lambda & \text{if } b \in (\bar{b}_\lambda, \bar{b}] \cap \overline{\Omega}; \end{cases}$$

(ii) The value function $V : \overline{\Omega} \to \mathbb{R}$ defined by $V(b) = \hat{V}(b)$ for $b \in \overline{\Omega}$;

(iii) The consumption policy function $C : \overline{\Omega} \to \mathbb{R}_+$ defined by

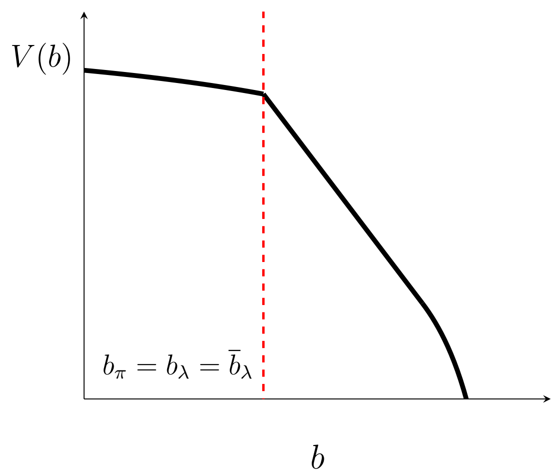
$$C(b) = \begin{cases} y - r^\star b & \text{if } b \in [b_{min}, \bar{b}_\lambda] \cap \overline{\Omega} \\ c_\lambda & \text{if } b \in (\bar{b}_\lambda, b^*] \cap \overline{\Omega} \\ y - (r^\star + \lambda)b & \text{if } b \in (b^*, b_{max}] \cap \overline{\Omega}; \end{cases}$$

(iv) The inflation policy function $\Pi : \overline{\Omega} \to \{0, \bar{\pi}\}$ defined by:

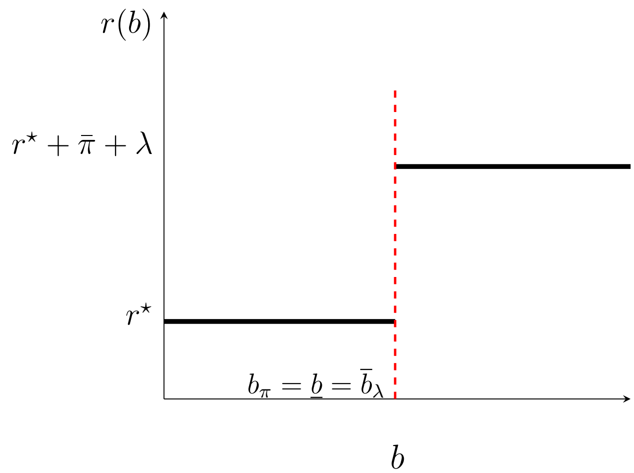
$$\Pi(b) = \begin{cases} 0 & \text{if } b \in [b_{min}, \bar{b}_\lambda] \cap \overline{\Omega} \\ \bar{\pi} & \text{if } b \in (\bar{b}_\lambda, b_{max}] \cap \overline{\Omega}. \end{cases}$$

This case is the mirror-image of case 2. In particular, the equilibrium crisis threshold and the inflation threshold are equivalent, but the reason is reversed. That is, the government increases inflation at $\bar{b}_\lambda$ because it faces a rollover crisis and wishes to reduce debt quickly.

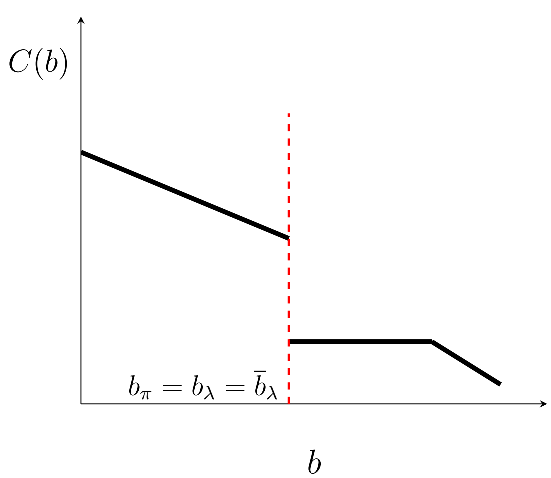
Figure 7: Case 3: Crisis Equilibrium if $\psi \in (\psi_2, \psi_3]$



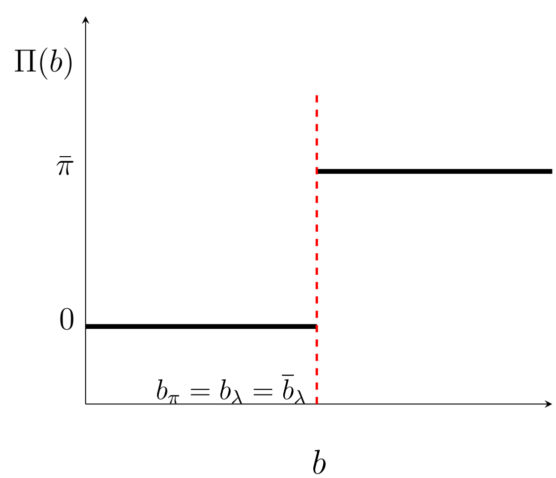
(a) Value Function



(b) Interest Rate



(c) Consumption Policy



(d) Inflation Policy

Therefore, the jump in interest rate due to a crisis leads the government to high inflation, rather than vice versa, as was the situation in case 2. Given this symmetry, the value function and policy functions in case 3 (figure 7) take the same form as those in case 2.

Case 4: $\psi_0 > \psi_3$

Proposition 7. *Suppose $\tilde{b}_\pi > \bar{b}_\lambda$ (that is, $\psi > \psi_3$). Define $c_\lambda \in (0, y - (r^\star + \lambda)\bar{b}_\lambda)$ as the unique solution to:*

$$\frac{(\rho+\lambda)u(y - r^\star \bar{b}_\lambda)}{\rho} = u(c_\lambda) - u'(c_\lambda)(c_\lambda - y + r^\star \bar{b}_\lambda) + \lambda \underline{V}.$$

Define $b^*_\lambda = (y - c_\lambda)/(r^\star + \lambda)$. For $b \leq \tilde{b}_\pi$, define $\hat{V}(b)$ by

$$\hat{V}(b) = \begin{cases} \frac{u(y - r^\star b)}{\rho} & \text{if } b \leq \bar{b}_\lambda \\ \frac{u(y - r^\star \bar{b}_\lambda)}{\rho} - u'(c_\lambda)(b - \bar{b}_\lambda) & \text{if } b \in (\bar{b}_\lambda, \min\langle b^*_\lambda, \tilde{b}_\pi \rangle) \\ \frac{u(y - (r^\star + \lambda)b)}{\rho + \lambda} + \frac{\lambda}{\rho+\lambda}\underline{V} & \text{if } b \in [b^*_\lambda, \tilde{b}_\pi]. \end{cases}$$

Define $c_\pi \in (0, y - (r^\star + \lambda)\tilde{b}_\pi)$ as the solution to

$$(\rho + \lambda)\hat{V}(\tilde{b}_\pi) = u(c_\pi) - \psi_0 \bar{\pi} - u'(c_\pi)(c_\pi + (r^\star + \lambda)\tilde{b}_\pi - y) + \lambda \underline{V}.$$

Let $b^*_\pi = (y - c_\pi)/(r^\star + \lambda)$. For $b > \tilde{b}_\pi$, define $\hat{V}(b)$ by

$$\hat{V}(b) = \begin{cases} V(\tilde{b}_\pi) - u'(c_\pi)(b - \tilde{b}_\pi) & \text{if } b \in (\tilde{b}_\pi, b^*_\pi) \\ \frac{u(y - (r^\star + \lambda)b) - \psi_0 \bar{\pi}}{\rho + \lambda} + \frac{\lambda}{\rho+\lambda}\underline{V} & \text{if } b \geq b^*_\pi. \end{cases}$$

Define $b_{max} = \max\{b \leq y/(r^\star + \lambda) | \underline{V} \leq \hat{V}(b)\}$. Then define $\overline{\Omega} = [b_{min}, b_{max}]$ for $b_{min} \in \mathbb{R}_-$, and the following constitutes a Recursive Equilibrium with Crisis:

(i) The interest rate schedule $r : \overline{\Omega} \to \{r^\star, r^\star + \lambda, r^\star + \bar{\pi} + \lambda\}$ defined by

$$r(b) = \begin{cases} r^\star & \text{if } b \in [b_{min}, \bar{b}_\lambda] \cap \overline{\Omega} \\ r^\star + \lambda & \text{if } b \in (\bar{b}_\lambda, \tilde{b}_\pi] \cap \overline{\Omega} \\ r^\star + \bar{\pi} + \lambda & \text{if } b \in (\tilde{b}_\pi, b_{max}] \cap \overline{\Omega}; \end{cases}$$

(ii) The value function $V : \overline{\Omega} \to \mathbb{R}$ defined by $V(b) = \hat{V}(b)$ for $b \in \overline{\Omega}$;

(iii) The consumption policy function $C : \overline{\Omega} \to \mathbb{R}_+$ defined by

$$C(b) = \begin{cases} y - r^\star b & \textit{if } b \in [b_{min}, \bar{b}_\lambda] \cap \overline{\Omega} \\ c_\lambda & \textit{if } b \in (\bar{b}_\lambda, \min\langle b^*_\lambda, \tilde{b}_\pi \rangle] \cap \overline{\Omega} \\ y - (r^\star + \lambda) b & \textit{if } b \in (b^*_\lambda, \tilde{b}_\pi] \cap \overline{\Omega} \\ c_\pi & \textit{if } b \in (\tilde{b}_\pi, b^*_\pi] \cap \overline{\Omega} \\ y - (r^\star + \lambda) b & \textit{if } b \in (b^*_\pi, b_{max}] \cap \overline{\Omega}; \end{cases}$$

(iv) The inflation policy function $\Pi : \overline{\Omega} \to \{0, \bar{\pi}\}$ defined by:

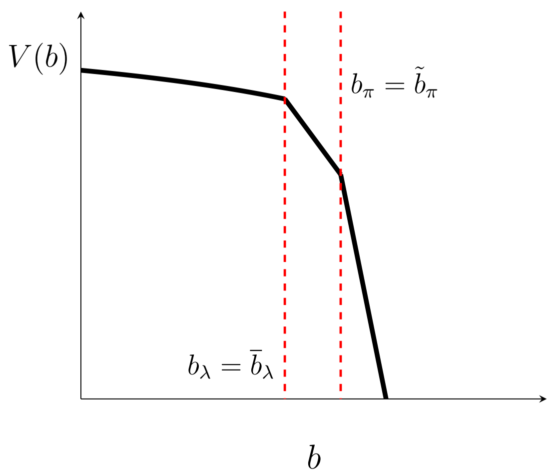
$$\Pi(b) = \begin{cases} 0 & \textit{if } \in b \in [b_{min}, \tilde{b}_\pi] \cap \overline{\Omega} \\ \bar{\pi} & \textit{if } b \in (\tilde{b}_\pi, b_{max}] \cap \overline{\Omega}. \end{cases}$$

Case 4 is an environment with a strong commitment to low inflation. It is optimal to set inflation to zero even in part of the crisis zone ($b \in (b_\lambda, \tilde{b}_\pi]$), despite the strong incentive to reduce debt in the neighborhood of b_λ. As $\psi_0 \to \infty$, $\tilde{b}_\pi \to \infty$, and there is zero inflation over the entire domain $\overline{\Omega}$ and in response to a rollover crisis. This corresponds to the environment of Cole and Kehoe (2000) in which debt is real, both on and off the equilibrium path. The value and policy functions depicted in figure 8 indicate the typical incentives to save at each increase in the interest rate, with the value function being linear in these regions.

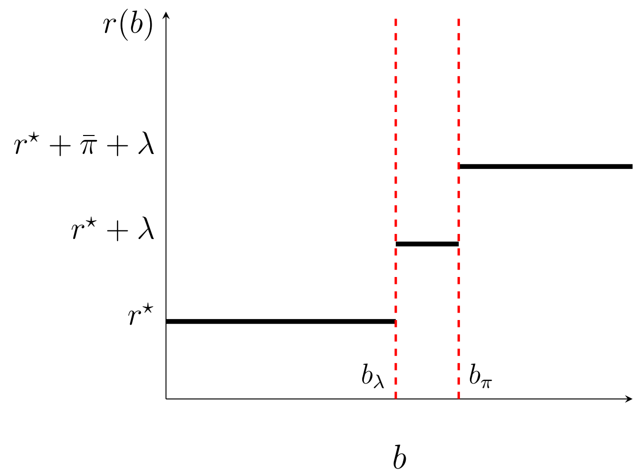
4.5 Inflation Commitment and Crisis Vulnerability

An important result depicted in figure 4 is that the extent of the safe zone is non-monotonic in ψ_0. In particular, the bold portion of panel (a) depicts the equilibrium b_λ; that is, the threshold for debt above which a rollover crisis can occur in equilibrium. The safe zone for government is $b \leq b_\lambda$. For low costs of inflation, in the regions surrounding ψ_1, the safe zone is strictly smaller than if $\psi_0 = \infty$. That is, issuing nominal bonds enlarges the range in which a rollover crises is possible relative to foreign currency bonds. The intuition is that the commitment to low inflation is so weak that the country faces a high nominal interest rate. This reduces the usefulness of inflating away debt in response to a crisis, as inflation is already priced in, making default more attractive. If ψ_0 is low but positive, the country still pays the cost of inflation in responding to a crisis, but gets no benefit relative to the equilibrium interest rate. This is why b_λ is declining in ψ_0 for $\psi_0 \in [0, \psi_1]$. The downward sloping curve traces out $\underline{b}_\lambda$, which reflects that W is decreasing in ψ_0 when the government

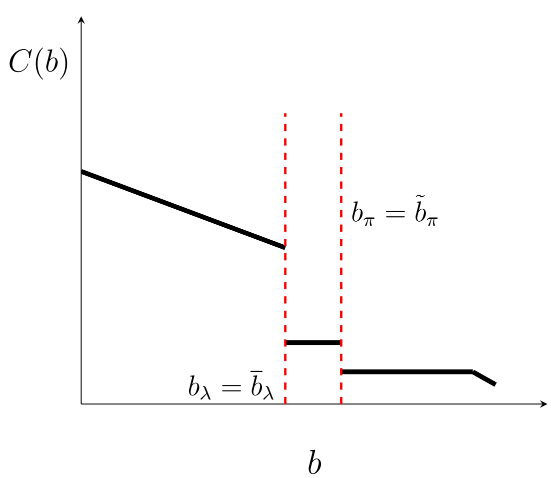
Figure 8: Case 4: Crisis Equilibrium if $\psi_0 > \psi_3$



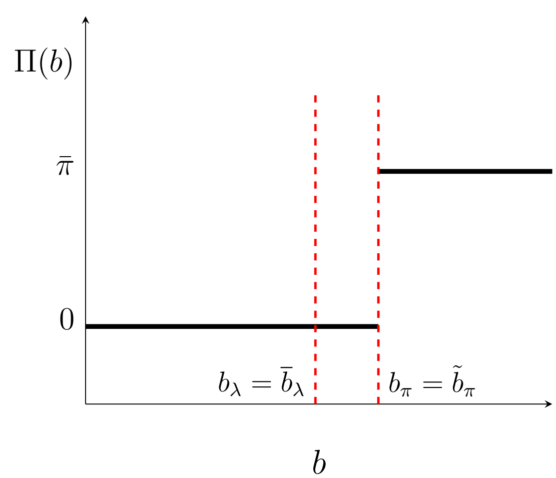
(a) Value Function



(b) Interest Rate



(c) Consumption Policy



(d) Inflation Policy

inflates in the grace period.[14]

The negative relationship between b_λ and ψ_0 is reversed once $\bar{b}_\pi = \underline{b}_\lambda$ (i.e., $\psi_0 = \psi_1$). At this debt level, the safe zone starts expanding with inflation commitment. This reflects the fact that the temptation to inflate absent a crisis creates the vulnerability to a crisis. The stronger the commitment to inflation in tranquil periods, the less vulnerable the economy is to a rollover crisis. At some threshold ψ^*, nominal bonds generate a larger safe zone. This is the happy medium in which inflation is not high in normal times, but the option to increase inflation in response to a crisis provides insurance. For ψ_0 above ψ^*, therefore, the economy can approximate state-contingent inflation relatively well. The safe zone peaks when $\psi_0 = \psi_3$, at which point the safe zone begins to shrink again. In this region, the costs of inflation not only reduce inflation in tranquil periods, but also make responding to a rollover crisis with inflation very costly. As ψ_0 becomes very large, the cost of inflation is so great that the government will not inflate even in a crisis. In the limit, the size of the safe zone converges to that of $\psi_0 = 0$, as in both cases the real value of bonds is independent of the arrival of a crisis.

This non-monotonicity of the safe zone with respect to inflation commitment is not due to the discreteness (or linear costs) of inflation choices. In numerical examples with strictly convex costs of inflation, the non-monotonicity in regard to the costs of inflation is verified. These simulations will be added in an appendix in a future version.

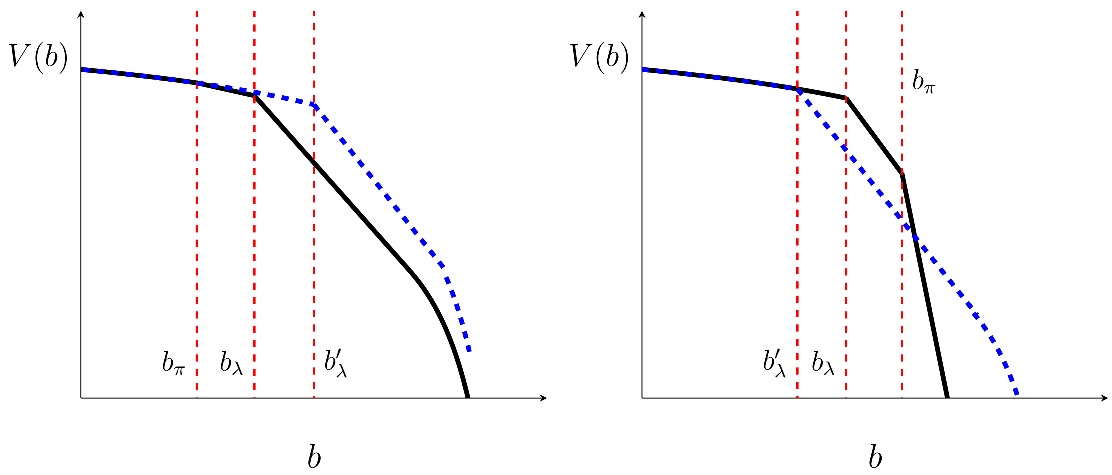
Moreover, the depiction makes clear that the slopes of $\underline{b}_\lambda$ and $\bar{b}_\lambda$ depend on the assumption that ψ_0 is independent of the arrival of a crisis. However, $\bar{b}_\pi$ depends on the non-crisis ψ_0. Therefore, even if $\underline{b}_\lambda$ and $\bar{b}_\lambda$ were independent of ψ_0 (that is, they depended on a crisis-specific cost of inflation), the fact remains that the equilibrium interest rate determines whether $\underline{b}_\lambda$ or $\bar{b}_\lambda$ is the relevant threshold, and this depends on the non-crisis ψ_0 implicit in $\bar{b}_\pi$.

A second question is whether an economy is better off issuing nominal or real debt. We depict two cases in figure 9. In each panel, the dashed line is the value function for $\psi_0 = \infty$, which corresponds to issuing foreign currency bonds. The solid line is the value from issuing nominal debt, where the two panels differ by the costs of inflation. All lines coincide for low b as inflation is zero and there is no risk of a crisis in this region.

Panel (a) is such that $\psi_0 \leq \psi^*$, so the safe zone is smaller with nominal bonds. In particular, b_π, the point at which the economy begins inflating, is within the safe zone. At this point, the nominal bond economy becomes worse off due to the inability to deliver

[14] If the grace period is long enough and ψ_0 high enough, the government may not inflate during the grace period. In this parameter space, $\underline{b}_\lambda$ and $\bar{b}_\lambda$ have slope zero. Figure 4 depicts the case in which the thresholds are decreasing at the points of intersection with the inflation cutoffs. The crisis thresholds are strictly decreasing at $\psi_0 = 0$, as inflation will always be optimal for low enough costs. The eventual flattening out of the crisis thresholds as $\psi_0 \to \infty$ is implied as $\bar{b}_\lambda$ converges to the horizontal dashed line in panel (a).

Figure 9: Government Welfare as a Function of Inflation Commitment



(a) $\psi_0 < \psi^*$ vs. $\psi_0 = \infty$

(b) $\psi_0 > \psi^*$ vs. $\psi_0 = \infty$

low inflation. At b_λ, the economy becomes vulnerable to a rollover crisis, while the crisis threshold is b'_λ for the foreign currency bond scenario. The safe zone is smaller with nominal bonds as debt carries with it the burden of inflation, making default relatively attractive. In this case, the economy is always strictly better off with foreign currency debt. The incentive to inflate is high in equilibrium, lowering welfare without reducing the exposure to a rollover crisis. Most emerging markets rely solely on foreign currency debt for international bond issues. The analysis rationalizes this so-called "original sin" as the optimal response to a weak inflationary regime, with or without self-fulfilling debt crises.

Panel (b) depicts a case in which $\psi_0 > \psi^*$. That is, nominal bonds reduce the exposure to a rollover crisis, but at the expense of higher equilibrium inflation for very large debt levels. This makes nominal bonds optimal for intermediate stocks of debt, but sub-optimal for high levels of debt. The closer ψ_0 is to the peak-safe-zone level ψ_3, the greater the range for which domestic currency debt strictly dominates. Thus governments that have a moderate degree of inflation commitment strictly prefer domestic currency debt over a non-negligible interval of debt. For extremely high levels of debt, the economy will inflate (and face a crisis), and so the commitment to zero inflation in this region is preferable.

5 Conclusion

In this paper we explored the role inflation commitment plays in vulnerability to a rollover crisis. We confirmed that for an intermediate level of inflationary commitment, an economy is less vulnerable to a crisis with nominal bonds. The intermediate commitment provides the missing state contingency, delivering low inflation in tranquil periods but high inflation in response to a crisis. Extreme commitment to low inflation eliminates the option to inflate in a crisis. In the model, strong commitment can be seen as equivalent to issuing foreign currency debt; such commitment may also arise by being a small member of a monetary union subject to idiosyncratic rollover risk. On the other hand, weak commitment to inflation renders an economy more vulnerable to a rollover crisis if it issues domestic currency bonds. This rationalizes the exclusive issuance of foreign currency bonds to international investors by governments with limited inflation credibility.

Appendices

A Proofs

Under Construction: All proofs are preliminary.

A.1 Proof of Proposition 1

Proof. Our model is a particular case of the general environment studied by Bressan and Hong (2007) (henceforth, BH). The proof therefore involves ensuring the hypotheses in BH are satisfied. We alter some of the BH notation to be consistent with our text, and translate the minimization of cost problem considered by BH into a maximization of utility. BH restrict attention to non-negative costs (non-positive utility), which we incorporate by re-defining $v(x) = v(x) - \bar{u}$ for all $x \in X$, where $\bar{u}$ is the upper bound on utility from consumption. BH consider the state space over the entire real line. We extend our problem to this larger domain by assigning the steady state utility to $b < b_{min}$ and some utility $\underline{u} \geq \rho \underline{V}$ for $b > b_{max}$, where $\underline{u}$ is chosen to ensure continuity of the value function at b_{max}. On these extended domains, we assume $f(b, x) = 0$ for all $x = X$, so there are no debt dynamics regardless of policy.We choose $\underline{u}$ to ensure continuity of the value function at b_{max}. As these domains have trivial decisions and dynamics, we do not explicitly discuss them in the verification of BH's hypotheses in what follows other than to include b_{min} and b_{max} as boundary points of discontinuous dynamics.

BH decompose the state space ($\mathbb{R}$ in our case) into $M < \infty$ disjoint manifolds (intervals in our case): $\mathbb{R} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup ... \cup \mathcal{M}_M$. In our environment, this corresponds to the points of discontinuity $\{b_{min}, b_1, ..., b_N, b_{max}\}$ as well as the intervening open sets, $(-\infty, b_{min}), (b_{min}, b_1), ..., (b_{max}, \infty)$. These satisfy the BH conditions: if $j \neq k$, then $\mathcal{M}_j \cap \mathcal{M}_k = \emptyset$; and if $\mathcal{M}_j \cap \overline{\mathcal{M}}_k \neq \emptyset$, then $\mathcal{M}_j \in \overline{\mathcal{M}}_k$. Let $i(b)$ denote the index of the interval that contains b.

Following BH, define a subset of controls $X_i \subset X$ for each interval $\mathcal{M}_i$ that produce tangent trajectories. That is,

$$X_i \equiv \left\{ x \in X \,\middle|\, \lim_{h \to 0} \frac{\inf_{b' \in \mathcal{M}_i} |b + f(b, x)h - b'|}{h} = 0 \,, \forall b \in \mathcal{M}_i \right\}.$$

Let $T_{M_i}(b)$ denote the set of feasible tangent trajectories for $b \in M_i$. For the open sets between points of discontinuity, all admissible controls produce tangent trajectories, and so $X_i = X$ and $T_{M_i}(b) = [\min_{x \in X} f(b, x), \max_{x \in X} f(b, x)]$. For the boundaries, $\{b_{min}, b_1, ..., b_{max}\}$, we have the steady state controls: $X_i = \{x | f(b_n, x) = 0\}$ if $\mathcal{M}_i = \{b_n\}$ and $T_{M_i}(b_n) = \{0\}$.

BH consider the following sets. Define

$$\hat{F}(b) \equiv \left\{(h,w)\,\middle|\,h = f(b,x), w \le v(b,x), x \in X_{i(b)}\right\} \subset \mathbb{R}^2.$$

This is the set of feasible tangent trajectories $f(b,x), x \in X_i$ paired with the payoff interval $(-\infty, v(b,x)]$. For a point b, we consider the convex combinations of tangent trajectories and associated utility in the neighborhood of b. In particular, let $\overline{co}S$ denote the convex hull of a set S. Define

$$G(b) \equiv \bigcap_{\epsilon>0} \overline{co} \left\{(h,l) \in \hat{F}(b') \,| |b'-b| < \epsilon\right\} \subset \mathbb{R}^2.$$

BH define the Hamilton-Jacobian-Bellman equation as:

$$\begin{gathered} \rho V(b) - \tilde{H}(b, V'(b)) = 0 \\ \text{where } \tilde{H}(b,q) \equiv \sup_{(h,w)\in G(b)} \{w + qh\}. \end{gathered} \tag{BH:HJB}$$

We now map the (BH:HJB) equation into our equation (HJB). Recall that

$$f_*(b,a) \equiv x - (r^\star + \pi)b - y + \lim \inf_{b' \to b} r(b')b'.$$

Similarly, define

$$f^*(b,a) \equiv x - (r^\star + \pi)b - y + \lim \sup_{b' \to b} r(b')b',$$

as the worst-case dynamics. Let $\mathcal{H}(b) \equiv [\min_{x\in X} f_*(b,x), \max_{x \in X} f^*(b,x)]$ as the relevant interval of debt dynamics for $x \in X$. Given b, and for $h \in \mathcal{H}(b)$, define

$$\hat{W}(h,b) = \max_{x\in X} v(b,x),$$

subject to $f_*(b,a) \le h$. $\hat{W}(h,b)$ represents the maximum utility of generating debt dynamics less than or equal to h. The function $h \mapsto \hat{W}(h,b)$ is non-decreasing and concave. We also have:

$$G(b) = \left\{(h,w)\,\middle|\,h \in \mathcal{H}(b), w \le \hat{W}(h,b)\right\}.$$

Moreover, for $q \leq 0$, we have

$$\tilde{H}(b,q) = \sup_{(h,w)\in G(b)} w + qh = \max_{x\in X} v(b,x) + qf_*(b,x) = H_*(b,q),$$

where H_* was defined in (3). With this equivalence, the definition of a viscosity solution given in the text corresponds to that used in BH.[15]

Given this mapping from our environment into that of BH, we now verify the BH assumptions. The definition of the set $\mathcal{R}$ ensure that conditions **H1** in BH hold on $\overline{\Omega}$, and our extension to the entire real line also satisfies the conditions on the extended domain. BH assumption **H2** holds in our environment as the tangent trajectories are either all trajectories (on the open sets of continuity) or the steady-state dynamics on the points of discontinuity. Condition **H3** in BH requires a weaker form of continuity than Lipschitz continuity, so our requirement of Lipschitz continuity for the viscosity solution satisfies this condition. Condition**H4** of BH requires that $V(b)$ is globally bounded. This is satisfied in our environment as $\frac{\bar{u}}{\rho} \geq V(b) \geq \underline{V}$ for all b. Finally, equation (46) in BH requires that the flow utility function be Lipschitz continuous with respect to b. As $v(x)$ is independent of b in our environment, this is satisfied trivially. Under these conditions, Corollary 1 in BH states that the value function is the unique viscosity solution to (HJB) satisfying these regularity properties. □

Proof of Lemma 1 and Proposition 2

We begin with the first claim in lemma 1:

Claim. *In any equilibrium, $r(b) \in \{r^\star, r^\star + \bar{\pi}\}$.*

Proof. Suppose not. Then there exists an open set (b', b'') such that $r(b) \in (r^\star, r^\star + \bar{\pi})$ for all $b \in (b_1, b_2)$. This follows from the lower semi-continuity requirement of equilibrium r.[16]Equilibrium requires that $\Pi(b) \in (0, \bar{\pi})$ for $b \in (b', b'')$. As V is Lipschitz continuous, it is differentiable almost everywhere. The optimization step in the Hamilton-Jacobi-Bellman equation implies that $-V'(b)b = u'(c)b = \psi_0$ for almost all $b \in (b', b'')$. Therefore $C(b)$ is increasing a.e. on (b', b''), which implies $C(b) \neq y - r^\star b$ a.e., which in turn implies that $f(b, (C(b), \Pi(b))) \neq 0, a.e.$ for $b \in (b', b'')$. That is, debt and consumption are not constant over time outside a set of measure zero for $b \in (b', b'')$. Recall as well that $r(b)$ is continuous almost everywhere. For some initial $b \in (b', b'')$, we can thus find a non-negligible interval

[15] BH define the concept of a viscosity solution in the context of a cost minimization problem. We redefine their definition to conform to a utility maximization problem.

[16] In particular, suppose the set $A \equiv \{b | r^\star < r(b) < r^\star + \bar{\pi}\}$ contained no open set; that is, it consisted of a finite set of points $\{b^1, b^2, ..., b^N\}$, then the set $\{b | r(b) > r^\star\}$ would be closed, contradicting lower semi-continuity.

of time $[0, \tau]$ such that $c(t)$ is not constant, $r(b(t))$ is continuous, but $r(b(t)) - \pi(t) = \rho$, a violation of optimality. Therefore, the candidate $r(b)$ cannot be an equilibrium. $\square$

The second part of lemma 1 concerns monotonicity:

Claim. *All equilibria are monotone. That is, $r(b)$ is non-decreasing in b.*

Proof. We proceed by considering a non-monotone $r(b)$, impose the equilibrium choice of inflation implied by $r(b)$, and solve for the government's optimal consumption. We then show that this generates a contradiction if $r(b)$ is not monotone.

Consider a non-monotone $r(b)$ with domain $\overline{\Omega}$. Specifically, let $I \equiv \{i | r(b) = r^\star + \bar{\pi}, \forall b \in \Omega_i\}$, denote the intervals for which $r(b) = r^\star + \bar{\pi}$. Equilibrium requires that $r(b) \in \{r^\star, r^\star + \bar{\pi}\}$ for all $b \in \overline{\Omega}$. Lower-semicontinuity of $r \in \mathcal{R}$ implies that high-interest domains are open sets. It is straightforward to show that $\Pi(b) = 0$ for $b \leq 0$ in any equilibrium, and so we can rule out $0 \in I$. Therefore, any non-monotone equilibria has $1 \in I$, implying that I is the set of odd integers less than or equal to N.

The proof proceeds by first characterizing the value function and consumption policy function on Ω_i, $i \in I$, imposing equilibrium conditions regarding the inflation policy function. We then derive a contradiction regarding optimal inflation policy. Let V denote our candidate equilibrium value function, and Π and C the corresponding policy functions. We impose that $\Pi(b) = \bar{\pi}$ for all $b \in \overline{\Omega}_i$, $i \in I$, and that $\Pi(b) = 0$ otherwise. This is a requirement of equilibrium.

We construct a candidate V as follows. Let $V(b) = u(y - r^\star b)/\rho$ for all $b \in \overline{\Omega}_j$, $j \notin I$. That is, when $r(b) = r^\star = \rho$, the optimal consumption policy is to set $\dot{b} = 0$. For $i \in I$, we construct the value function piecewise on the domain $\Omega_i = (b_i, b_{i+1})$. We consider the case of $i < N$ first; that is, intervals of high interest that do no include the upper bound debt m. This case is depicted in figure A1.

Starting from the upper end point of (b_i, b_{i+1}), let $c_{i+1}^- > y - r^\star b_{i+1}$ solve

$$\rho V(b_{i+1}) = u(c_{i+1}^-) - \psi_0 \bar{\pi} - u'(c_{i+1}^-) \left(c_{i+1}^- + r^\star b_{i+1} - y \right).$$

The "$-$" reflects that we are considering a neighborhood to the left of b_{i+1}. Note that $V(b_{i+1})$ is the low-interest, low-inflation steady state, and the HJB that defines c_{i+1}^- imposes the equilibrium condition $\pi = \bar{\pi}$. Define $V_{i+1}(b) = V(b_{i+1}) - u'(c_{i+1}^-)(b - b_{i+1})$ for $b \in [(y - c_{i+1}^-)/r^\star, b_{i+1})$ and $V_{i+1}(b) = (u(y - r^\star b) - \psi_0 \bar{\pi})/\rho$ for $b \in (b_i, (y - c_{i+1}^-)/r^\star)$. Let $C_{i+1}(b)$ be the consumption policy function associated with V_{i+1}. Note that by construction c_{i+1}^- satisfies the HJB at b_{i+1} with $\pi = \bar{\pi}$. In particular, it is the solution that implies borrowing towards the low-interest zone $\overline{\Omega}_{i+1}$. This function is depicted as V_{i+1} in figure A1, panel (a), in the case that there is no steady state between b_i and b_{i+1}. The consumption policy function c^-_{i+1} is depicted as in panel (c).

Turning to the neighborhood above b_i, let $c^+_i \in (0, y - r^\star b_i)$ solve

$$\rho V(b_n) = u(c^+_i) - \psi_0 \bar{\pi} - u'(c^+_i)\left(c^+_i + r^\star b_i - y\right).$$

The "+" reflects that this will be optimal consumption in the neighborhood above b_n. Define $V_i(b) = V(b_i) - u'(c^-_i)(b - b_i)$ for $b \in (b_i, (y - c^+_i)/r^\star]$ and $V_i(b) = (u(y - r^\star b) - \psi_0\bar{\pi})/\rho$ for $b \in ((y - c^+_i)/r^\star, b_{i+1})$. Let $C_i(b)$ be the consumption policy function associated with V_i. Note that by construction c^+_i satisfies the HJB at b_i with $\pi = \bar{\pi}$. In particular, it is the solution that implies saving towards the low-interest zone Ω_{i-1}. (See V_i in figure A1 panel (a) and c^+_1 in panel (c)).

Note that $V'_i(b) < V'_{i+1}(b)$. Moreover, there exists $\tilde{b} \in \Omega_i$ such that $V_i(\tilde{b}) = V_{i+1}(\tilde{b})$. To see this, note that $V_i(b_i) = u(y - r^\star b_i)/\rho$. Moreover for $b \in \Omega_i$, $V'_i(b) = -u'(C_i(b)) \leq u'(y - r^\star b)$, as $C_i(b) \leq y - r^\star b$, with the inequality strict in the neighborhood of b_i. This implies that $V_i(b_{i+1}) < V(b_{i+1}) = V_{i+1}(b_{i+1})$. Similarly, $C_{i+1}(b) \geq y - r^\star b$ for $b \in \Omega_i$. This implies that $V'_{i+1}(b) \geq u'(y - r^\star b)$, with the inequality strict in the neighborhood of b_{i+1}. As $V_{i+1}(b_{i+1}) = u(y - r^\star b_{i+1})/\rho$, we have that $V_{i+1}(b_i) < V(b_i) = V_i(b_i)$. By continuity, the two curves V_i and V_{i+1} must intersect in the interior of Ω_i.

Our candidate value function becomes $V(b) = V_i(b)$ for $b \in (b_i, \tilde{b}]$ and $V(b) = V_{i+1}(b)$ for $b \in (\tilde{b}, b_{i+1})$. The consumption policy function is defined accordingly. We can repeat these steps for all $i \in I$ such that $i < N$. That is, for all high-interest zones excluding $(b_N, m]$, where m is the upper bound on equilibrium debt.

If $N \in I$, that is, the final segment $(b_N, m]$ is also a high-interest rate zone, we proceed as follows. c^-_{N+1} is the solution to the HJB at $b = m$ replacing $V(m)$ with $(u(y - r^\star m) - \psi_0\bar{\pi})/\rho$, the high-inflation, high-interest steady state value function. The segment $v_{N+1}(b)$ is constructed accordingly. The segment $v_N(b)$ is constructed as before, by picking the saving solution to the HJB at $b = b_N$. However, there is no guarantee that $v_N(m) \leq v_{N+1}(m)$, as the latter is the high-inflation steady state value function and it may be optimal to save towards b_N from all $b \in \Omega_N$. If $v_N(m) \leq v_{N+1}(m)$, there exists an intersection point $\tilde{b}$ and we proceed as before. If not, then $V(b) = v_N(b)$ for all $b \in (b_N, m]$.

The value function $V(b)$ so constructed is a viscosity solution to HJB,assuming the policy function $\Pi = r(b) - r^\star$ implied by equilibrium is indeed optimal. It is therefore the only possible value function consistent with equilibrium. The contradiction arises as follows. Note that $C(b_{i+1}) = y - r^\star b_{i+1}$, for $i \in I$ and $i < N$, consistent with the low-interest, low-inflation steady state value at the end point of a high inflation zone. Optimality of inflation requires that $u'(y - r^\star b_{i+1})b_{i+1} \leq \psi_0$. However, $\lim_{b \uparrow b_{i+1}} C(b) = c^-_{i+1} > y - r^\star b_{i+1}$, the latter inequality following from the definition of c^-_{i+1} as the borrowing solution to the HJB at b_{i+1}. Optimality of high inflation as we approach b_{i+1} from below requires that $u'(c^-_{i+1})b_{i+1} < \psi_0$. The combined implication that $\psi_0 \leq u'(y - r^\star b_{i+1})b_{i+1} < u'(c^-_{i+1})b_{i+1} < \psi_0$ generates the contradiction.

□

The proof of proposition 2:

Proof. The proposition characterizes by construction all equilibria with $b_\pi \in [\underline{b}_\pi, \overline{b}_\pi]$. Equilibria for b_π outside this interval can be ruled out using the definition of the intervals. In particular, equilibrium requires that $\Pi(b) = r(b) - r^\star$. Impose this condition on the government's problem and solve for optimal consumption. At b_π, implied inflation is zero and $r(b) = r^\star = \rho$. The government's optimal policy response is to set $C(b_\pi) = y - r^\star b_\pi$, so that $\dot{b} = 0$ and $V(b_\pi) = u(y - r^\star b)/\rho$. We now check whether consumption is consistent with implied inflation using the HJB equation at b_π. Optimal consumption in the neighborhood above b_π is given by $C_\pi(b_\pi)$ from equation (4). If $b_\pi < \underline{b}_\pi$, this consumption is inconsistent with high inflation, violating the equilibrium requirement to the right of b_π. Conversely, if $b_\pi > \overline{b}_\pi$, then zero inflation is inconsistent with the steady state consumption at b_π, violating the equilibrium requirement that $\Pi(b_\pi) = 0$. □

Proof of Proposition 3

Proof. The proof follows directly from Bressan and Hong (2007). See the proof of Proposition 1 for details. □

Proof of Lemma 2

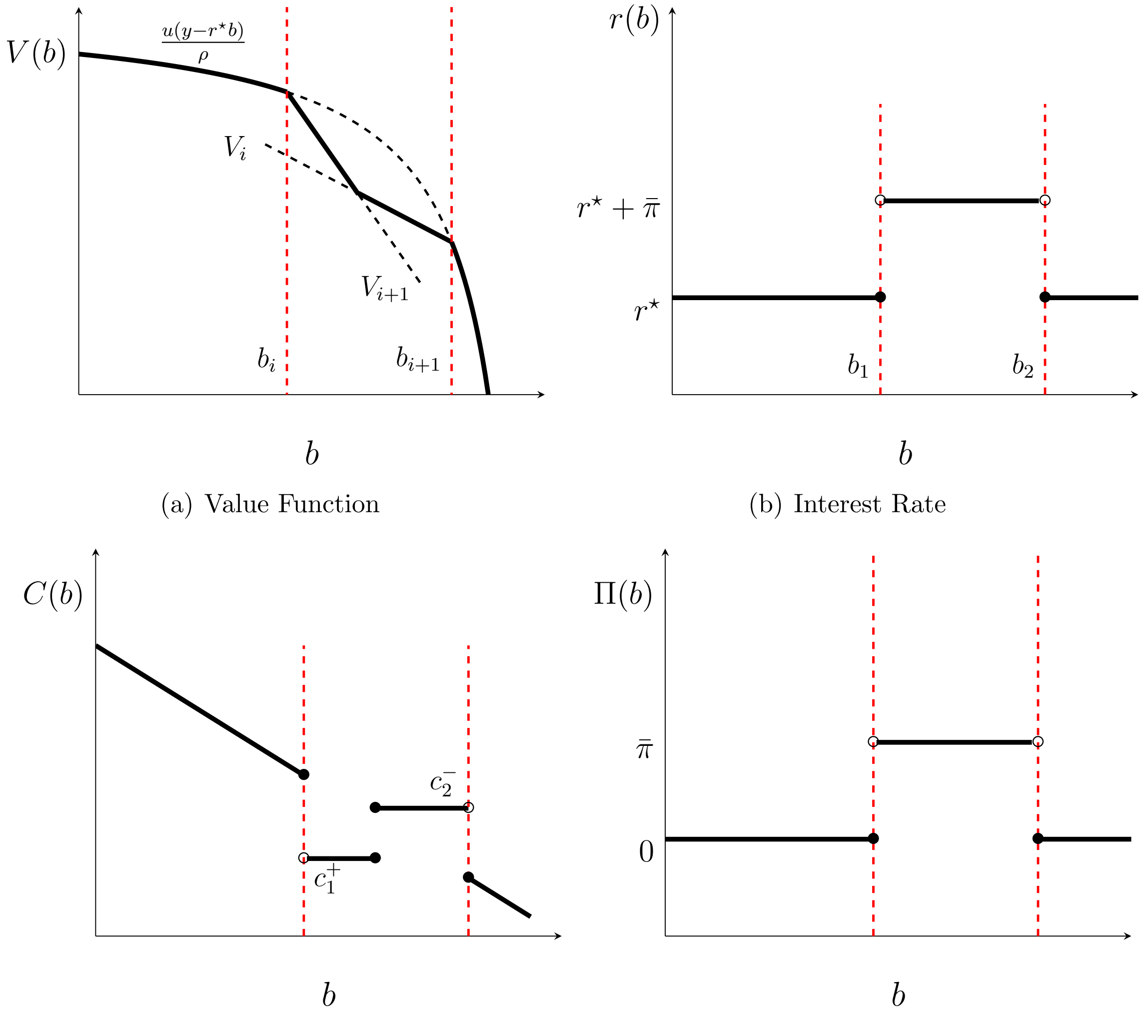
Proof. The proof of this lemma is the same as the proof of the part of lemma 1. Namely, $\Pi(b) = \{0, \bar{\pi}\}$. □

Proofs of Propositions 4, 5, 6 and 7

Proof. These propositions follow by construction. □

B Convex Inflation Costs

Appendix Figure A1: Government's Solution with No Crisis: Non-Monotone $r(b)$



(a) Value Function

(b) Interest Rate

(c) Consumption Policy

(d) Implied Inflation Policy $(r(b) - r^\star)$

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