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SOLVING LINEAR RATIONAL EXPECTATIONS MODELS WITH PREDICTABLE STRUCTURAL CHANGE

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Abstract

Standard solution methods for linear stochastic models with rational expectations presuppose a time-invariant structure as well as an environment in which shocks are unanticipated. Consequently, credible announcements that entail future changes of the structure cannot be handled by standard solution methods. This paper develops the solution for linear stochastic rational expectations models in the face of a finite sequence of anticipated structural changes. These events encompass anticipated changes to the structural parameters and anticipated additive shocks. We apply the solution technique to some examples of practical relevance to monetary policy.

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1. Introduction

Methods for solving linear stochastic models with rational expectations, like Anderson (1997), Blanchard and Kahn (1980), Binder and Pesaran (1995), Sims (2002), Klein (2000) and Uhlig (1995), assume a time-invariant structure; the parameters that govern the behaviour of the system are taken to be constant. Although the rational expectations solution has recently been extended so as to allow some of the parameters to vary in accordance with an exogenous Markov process with given transition probabilities – see Davig and Leeper (2007) and Farmer, Waggoner and Zha (2007) – these methods do not handle credible announcements that entail future variations to the structural parameters.

The situations we have in mind are not merely theoretical curiosities, but rather real situations of practical importance. Take for instance the case of Chile with respect to announcements regarding inflation targets. For example, the first inflation target was announced in September 1990 for the 12 months of 1991; later, in September 1999 the central bank announced a point target for 2000 and also, starting in 2001, a stationary target range for the indefinite future.[1] Other examples include the announcement of the introduction of the goods and services tax in Australia in 2000 and the recent announcement by the UK Government to lower the VAT only to increase it again after 13 months. To the extent that such announcements are credible, the behaviour of the economy in the periods between the announcement of the policy and its implementation would be poorly captured using available solution methods.

As emphasised by Marschak (1953), in the case of a foreseen change in structure, the purely empirical projection of observed past regularities into the future cannot be used reliably in decision-making. To produce meaningful forecasts, knowledge of the past structure and of observed past regularities has to be supplemented by the way the structure is expected to change.

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[1] See Morandé and Schmidt-Hebbel (2000)

This paper establishes a rational expectations solution for linear stochastic models in the face of predictable structural variations. The next section reviews the time-invariant solution of Sims (2002) for linear rational expectations models upon which we build to develop the solution under anticipated structural variations. Section 3 states the problem formally and then develops the rational expectations solution under predictable structural and additive variations. Section 4 illustrates the solution with a set of numerical examples while Section 5 concludes.

2.   The Time-invariant Rational Expectations Solution

The method to solve for equilibria in linear rational expectations (LRE) models with predictable structural variations builds on the method proposed in Sims (2002). We begin by introducing notation, then outline the solution in the time-invariant case, and establish key results on existence and uniqueness.

2.1   Defining the Linear Rational Expectations Model

Define the state vector

$$y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \\ \mathbb{E}_t z_{t+1} \end{pmatrix}$$

where: $y_{1,t}$, $(n_1 \times 1)$, contains exogenous and possibly some endogenous variables; $y_{2,t}$, $(n_2 \times 1)$, contains those endogenous variables for which conditional expectations appear in the LRE model; and z_{t+1}, $(k \times 1)$, contains leads of $y_{2,t}$ so that $z_{t+1} = (y'_{2,t+1}, \ldots, y'_{2,t+s})'$ and $k = s \times n_2$. The dimension of y_t is $n \times 1$, where $n = n_1 + n_2 + k$.

The LRE model is typically given by $n_1 + n_2$ equations relating the elements of $y_{1,t}$ and $y_{2,t}$ to each other and to $\mathbb{E}_t z_{t+1}$

$$\tilde{\Gamma}_0 y_t = \tilde{\Gamma}_1 y_{t-1} + \tilde{C} + \tilde{\Psi} \varepsilon_t \tag{1}$$

where: ε_t is a $l \times 1$ vector that is a random, exogenous and potentially serially correlated process; $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$ are $(n_1 + n_2) \times n$ matrices; $\tilde{C}$ is $(n_1 + n_2) \times 1$; and $\tilde{\Psi}$ is $(n_1 + n_2) \times l$.

Since we allow z_{t+1} to potentially contain more than just one lead of $y_{2,t}$, we deviate from the terminology of Sims (2002) and define the vector of expectations

revisions as follows

$$\eta_t = \mathbb{E}_t z_t - \mathbb{E}_{t-1} z_t \tag{2}$$

where $\mathbb{E}_t \eta_{t+j} = 0$ for $j \geq 1$. When $z_t = y_{2,t}$, η_t becomes a vector of forecast errors $(\eta_t = y_{2,t} - \mathbb{E}_{t-1} y_{2,t})$. Note that $\mathbb{E}_t z_t = (y'_{2,t}, \mathbb{E}_t y_{2,t+1}, \ldots, \mathbb{E}_t y_{2,t+s-1})'$ so $\mathbb{E}_t z_t$ incorporates $y_{2,t}$ and the first $(s-1)$ elements of $\mathbb{E}_t z_{t+1}$. So expectation revisions for $y_{2,t+s}$ do not appear in Equation (2). It is also important to note that the information set in period t contains the value of all variables up to period $t-1$ as well as period t shocks.

We augment the system defined by Equation (1) with the k equations from Equation (2) to obtain the following specification

$$
\begin{pmatrix} \tilde{\Gamma}_0 \\ 0_{k \times n_1} \quad I_k \quad 0_{k \times n_2} \end{pmatrix}
\begin{pmatrix} y_{1,t} \\ y_{2,t} \\ \mathbb{E}_t z_{t+1} \end{pmatrix}
= \begin{pmatrix} \tilde{C} \\ 0 \end{pmatrix}
$$

$$
+ \begin{pmatrix} \tilde{\Gamma}_1 \\ 0_{k \times (n_1+n_2)} \quad I_k \end{pmatrix}
\begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ \mathbb{E}_{t-1} z_t \end{pmatrix} \tag{3}
$$

$$
+ \begin{pmatrix} \tilde{\Psi} \\ 0 \end{pmatrix} \varepsilon_t
+ \begin{pmatrix} 0 \\ I_k \end{pmatrix} \eta_t
$$

which is equivalent, in the notation of Sims (2002), to

$$\Gamma_0 y_t = C + \Gamma_1 y_{t-1} + \Psi \varepsilon_t + \Pi \eta_t \tag{4}$$

where the matrices Γ_0 and Γ_1 are both $n \times n$, while C is $n \times 1$, Ψ is $n \times l$, and Π is $n \times k$. This system contains n equations – the same number as the number of variables in the state vector, y_t. It is worth noting that the vector of expectations revisions, η_t, is determined endogenously as part of the solution.

2.2 Solving the LRE Model

The Generalised Schur (or QZ) decomposition of (Γ_0, Γ_1) yields

$$
\begin{aligned}
Q' \Lambda Z' &= \Gamma_0 \\
Q' \Omega Z' &= \Gamma_1
\end{aligned}
$$

where $QQ' = ZZ' = I$ and both Λ and Ω are upper triangular. Q, Z, Λ and Ω are, in general, complex valued. An important property of this decomposition, which always exists, is that it returns the generalised eigenvalues of (Γ_0, Γ_1) as the ratios of the diagonal elements of Ω and Λ, $\{\omega_{ii}/\lambda_{ii}\}$.

Premultiply Equation (4) by Q to get

$$\Lambda w_t = \Omega w_{t-1} + Q(C + \Psi \varepsilon_t + \Pi \eta_t) \tag{5}$$

where $w_t = Z' y_t$. Then rearrange the system so that the explosive eigenvalues correspond to the lower right blocks of Λ and Ω and partition w_t as follows

$$w_t = \begin{pmatrix} Z_1' y_t \\ Z_2' y_t \end{pmatrix} = \begin{pmatrix} w_{1,t} \\ w_{2,t} \end{pmatrix}$$

where $w_{2,t}$ is a $m \times 1$ vector that is associated with the m explosive generalised eigenvalues and $w_{1,t}$ is $(n-m) \times 1$.

According to this partition, Equation (5) reads

$$
\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{pmatrix} \begin{pmatrix} w_{1,t} \\ w_{2,t} \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{pmatrix} \begin{pmatrix} w_{1,t-1} \\ w_{2,t-1} \end{pmatrix} \\
+ \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} (C + \Psi \varepsilon_t + \Pi \eta_t) \tag{6}
$$

As the lower set of equations is not influenced by $w_{1,t}$, the dynamics of $w_{2,t}$ are isolated as follows

$$\Lambda_{22} w_{2,t} = \Omega_{22} w_{2,t-1} + Q_2(C + \Psi \varepsilon_t + \Pi \eta_t) \tag{7}$$

Let $M \equiv \Omega_{22}^{-1} \Lambda_{22}$ and let $x_{2,t} \equiv Q_2(C + \Psi \varepsilon_t + \Pi \eta_t)$. Since the eigenvalues of Equation (7) are explosive, the equation can be solved forwards

$$
\begin{aligned}
w_{2,t} &= M w_{2,t+1} - \Omega_{22}^{-1} x_{2,t+1} \\
&= M^2 w_{2,t+2} - M \Omega_{22}^{-1} x_{2,t+2} - \Omega_{22}^{-1} x_{2,t+1} \\
&= \dots \\
&= -\sum_{j=1}^{\infty} M^{j-1} \Omega_{22}^{-1} x_{2,t+j}
\end{aligned}
$$

assuming $\lim_{j\to\infty} M^j w_{2,t+j} = 0$. Substituting back the definitions for M and $x_{2,t}$ and expanding the expression above for $w_{2,t}$ yields

$$w_{2,t} = (\Lambda_{22} - \Omega_{22})^{-1} Q_2 C - \sum_{j=1}^{\infty} M^{j-1} \Omega_{22}^{-1} Q_2 (\Psi \varepsilon_{t+j} + \Pi \eta_{t+j}) \qquad (8)$$

Equation (8) relates $w_{2,t}$ to future values of ε_t and η_t. This means that knowing $w_{2,t}$ requires that all future events be known at time t. Taking expectations (conditional on time t information) does not change the left-hand side of Equation (8), so

$$w_{2,t} = (\Lambda_{22} - \Omega_{22})^{-1} Q_2 C - \mathbb{E}_t \sum_{j=1}^{\infty} M^{j-1} \Omega_{22}^{-1} Q_2 \Psi \varepsilon_{t+j} \qquad (9)$$

since $\mathbb{E}_t \eta_{t+j} = 0$ for $j \geq 1$. The fact that the right-hand side of Equation (8) never deviates from its expected value implies that expectations revisions must fluctuate as a function of current and future ε_t's to guarantee that the equality holds.

Taking expectations at time $t+1$ gives

$$w_{2,t} = (\Lambda_{22} - \Omega_{22})^{-1} Q_2 C - \mathbb{E}_{t+1} \sum_{j=1}^{\infty} M^{j-1} \Omega_{22}^{-1} Q_2 \Psi \varepsilon_{t+j} - \Omega_{22}^{-1} Q_2 \Pi \eta_{t+1} \qquad (10)$$

Expressions (9) and (10) are equal if and only if the vector of expectations revisions satisfies

$$Q_2 \Pi \eta_{t+1} = \Omega_{22} \sum_{j=1}^{\infty} M^{j-1} \Omega_{22}^{-1} Q_2 \Psi (\mathbb{E}_t \varepsilon_{t+j} - \mathbb{E}_{t+1} \varepsilon_{t+j}) \qquad (11)$$

The system's stability depends on the existence of expectations revisions η_t to offset the effect that the fundamental shocks ε_t have on $w_{2,t}$. To see this, assume $\mathbb{E}_t \varepsilon_{t+i} = 0$ for $i \geq 1$ and $C = 0$. The equation for $w_{2,t}$ becomes

$$w_{2,t} = \Lambda_{22}^{-1} \Omega_{22} w_{2,t-1} + \Lambda_{22}^{-1} Q_2 (\Psi \varepsilon_t + \Pi \eta_t) \qquad (12)$$

Since this equation has explosive eigenvalues, stability requires that $w_{2,t} = 0$ for all t. This means that $Q_2 \Psi \varepsilon_t + Q_2 \Pi \eta_t = 0$ must hold in each period to ensure that the

effect on $w_{2,t}$ of any fundamental shock (ε_t) is offset by revisions to expectations, η_t; if this condition does not hold, $w_{2,t}$ will behave explosively.

The existence of a stable solution relies on expectations revisions (η_t) to adjust so that the system remains on its stable saddle path (SSP). This means that from any arbitrary starting point, expectations revisions must be able to get the system onto its SSP and then keep it there. Proposition 1 states the condition under which this is possible.

Proposition 1. *For any initial starting value y_0, a stable solution exists for the following linear rational expectations system*

$$\Gamma_0 y_t = C + \Gamma_1 y_{t-1} + \Psi \varepsilon_t + \Pi \eta_t$$

if and only if $rank(Q_2\Pi) = m$.

For a proof of Proposition 1, see Appendix A.

Since $Q_2\Pi$ is $m \times k$, $rank(Q_2\Pi) \leq \min\{m, k\}$, so the existence of a stable solution requires that $m \leq k$; that is, the number of explosive eigenvalues cannot be larger than the dimension of η_t.

Proposition 1 states the conditions for existence with arbitrary initial conditions. Should the system already be on its stable saddle path, the rank condition is only sufficient for existence. If initial conditions place the system on its SSP, then the conditions for existence of a stable solution are weaker. Existence, in this case, requires that there is a vector of expectations revisions capable of offsetting the effect of new information on $w_{2,t}$. For this to occur, it is both necessary and sufficient that

$$span\left(\left\{\Omega_{22}M^{j-1}\Omega_{22}^{-1}Q_2\Psi\right\}_{j=1}^{m}\right) \subseteq span(Q_2\Pi) \tag{13}$$

Regardless of what process ε_t follows, the existence of a rational expectations solution requires solving a system of the form: $Q_2\Pi\eta_t = B_t$, where $Q_2\Pi \in \mathbb{C}^{m \times k}$, $\eta_t \in \mathbb{R}^k$ and $B_t \in \mathbb{C}^m$. The span condition is both necessary and sufficient for the vector B_t to be expressed as a linear combination of the columns of $Q_2\Pi$ and guarantees that a solution exists for η_t.

The kind of parameter variations that we consider in the next section typically alter the SSP of the system. Therefore, it is the rank condition that ensures stability. Announcements about future changes to the structure give rise not only to changes to the SSP, but also to arbitrary 'initial conditions' from the perspective of the new SSP. Although the span and rank conditions for existence of a stable solution would typically agree, it is the rank condition which is appropriate if initial conditions are indeed arbitrary.

Existence does not imply uniqueness. In general, it is possible that knowing $Q_2 \Pi \eta_t$ may not be enough to calculate $Q_1 \Pi \eta_t$, which is needed in order to solve for $w_{1,t}$ and to completely solve the LRE model. This requires that the row space of $Q_1 \Pi$ be contained in the row space of $Q_2 \Pi$, both of which are subspaces of $\mathbb{R}^k$. It turns out that checking the row span condition for the uniqueness of an equilibrium is equivalent to checking the rank of the matrix $Q_2 \Pi$ as the following proposition states.

Proposition 2. *Suppose a solution exists for the following linear rational expectations system*

$$\Gamma_0 y_t = C + \Gamma_1 y_{t-1} + \Psi \varepsilon_t + \Pi \eta_t$$

Then the solution is unique if and only if $rank(Q_2 \Pi) = k$.

For a proof of Proposition 2, see Appendix A.

Since $rank(Q_2 \Pi) \leq \min\{m, k\}$, this implies that $m \geq k$ is a necessary condition for a unique solution. For arbitrary initial conditions, existence and uniqueness of a solution requires that $m = k$.

If a unique solution exists, then there exists a matrix Φ such that

$$Q_1 \Pi = \Phi Q_2 \Pi \tag{14}$$

Pre-multiplying equation (6) by $[I_{n-m}, -\Phi]$ yields

$$\begin{aligned}
\begin{pmatrix} \Lambda_{11} & \Lambda_{12} - \Phi \Lambda_{22} \end{pmatrix} & \begin{pmatrix} w_{1,t} \\ w_{2,t} \end{pmatrix} \\
&= \begin{pmatrix} \Omega_{11} & \Omega_{12} - \Phi \Omega_{22} \end{pmatrix} \begin{pmatrix} w_{1,t-1} \\ w_{2,t-1} \end{pmatrix} + (Q_1 - \Phi Q_2) C \\
&\quad + (Q_1 - \Phi Q_2) \Psi \varepsilon_t + (Q_1 \Pi - \Phi Q_2 \Pi) \eta_t
\end{aligned} \tag{15}$$

When such a Φ exists, the term involving η_t drops out. Combining Equations (15) and (9), it is not difficult to show that the reduced-form of the LRE model becomes

$$y_t = S_0 + S_1 y_{t-1} + S_2 \varepsilon_t + S_y \mathbb{E}_t \sum_{j=1}^{\infty} M^{j-1} \Omega_{22}^{-1} Q_2 \Psi \varepsilon_{t+j} \qquad (16)$$

where

$$H = Z \begin{pmatrix} \Lambda_{11}^{-1} & -\Lambda_{11}^{-1}(\Lambda_{12} - \Phi\Lambda_{22}) \\ 0 & I \end{pmatrix}; \quad S_0 = H \begin{pmatrix} Q_1 - \Phi Q_2 \\ (\Lambda_{22} - \Omega_{22})^{-1} Q_2 \end{pmatrix} C;$$

$$S_1 = H \begin{pmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{12} \\ 0 & 0 \end{pmatrix} Z'; \qquad S_2 = H \begin{pmatrix} Q_1 - \Phi Q_2 \\ 0 \end{pmatrix} \Psi; \text{ and}$$

$$S_y = -H \begin{pmatrix} 0 \\ I_m \end{pmatrix}$$

3. The Rational Expectations Solution with Predictable Structural Variations

In this section, we propose a method to solve LRE models when there is a sequence of anticipated events. These events encompass anticipated changes to the structural parameters of the model or anticipated additive shocks. We assume that within a finite period of time, the structural parameters of the model converge and no further shocks are anticipated.

At the beginning of period 1, agents know the previous state of the economy y_0, the fundamental shock ε_1, they anticipate a sequence of shocks $\{\varepsilon_t^a\}_{t=2}^T$, and know how the structural parameters will vary in the future, $\{\tilde{C}_1, \tilde{\Gamma}_{0,1}, \tilde{\Gamma}_{1,1}, \tilde{\Psi}_1, \Pi, \{C_t, \Gamma_{0,t}, \Gamma_{1,t}, \Psi_t\}_{t=2}^T, (\bar{C}, \bar{\Gamma}_0, \bar{\Gamma}_1, \bar{\Psi}, \bar{\Pi})\}$. That is, the system evolves as follows

$$
\begin{array}{ll}
\tilde{\Gamma}_{0,1} y_1 = \tilde{C}_1 + \tilde{\Gamma}_{1,1} y_0 + \tilde{\Psi}_1 \varepsilon_1 & t = 1 \\
\Gamma_{0,t} y_t = C_t + \Gamma_{1,t} y_{t-1} + \Pi \eta_t + \Psi_t(\varepsilon_t^u + \varepsilon_t^a) & 2 \leq t \leq T \\
\bar{\Gamma}_0 y_t = \bar{C} + \bar{\Gamma}_1 y_{t-1} + \bar{\Pi} \eta_t + \bar{\Psi} \varepsilon_t & t \geq T+1
\end{array} \qquad (17)
$$

where ε_t^u represents unanticipated shocks to the system and $\mathbb{E}_t \varepsilon_{t+j}^u = 0$ for $j \geq 1$. The reason for identifying these shocks separately is because as time unfolds,

actual shocks may be different from what was originally expected so that in any period, we can decompose a shock as the sum of its anticipated and unanticipated components $\varepsilon_t = \varepsilon_t^a + \varepsilon_t^u$. We could alternatively include $\Psi_t \varepsilon_t^a$ as part of C_t, but we identify the shocks separately to illustrate how the solution for predictable structural variations encompasses anticipated additive shocks as a special case.

Assuming a unique solution exists for $t \geq T+1$, the reduced form of the system can be computed as discussed in the previous section

$$y_t = \bar{S}_0 + \bar{S}_1 y_{t-1} + \bar{S}_2 \varepsilon_t + \bar{S}_y \mathbb{E}_t \sum_{j=1}^{\infty} \bar{M}^{j-1} \bar{\Omega}_{22}^{-1} \bar{Q}_2 \Psi \varepsilon_{t+j} \quad t \geq T+1 \qquad (18)$$

where $\bar{M} = \bar{\Omega}_{22}^{-1} \bar{\Lambda}_{22}$. This solution helps us compute y_t for $t \geq T+1$, given y_T. The aim of this section is to solve for $y_1, y_2, \ldots, y_T$ given all anticipated structural variations and additive shocks.

Since y_t is $(n_1 + n_2 + k) \times 1$, we require at least $T \times (n_1 + n_2 + k)$ independent equations to obtain a unique solution for $\{y_t\}_{t=1}^{T}$. Notice that:

- for each period, we have $(n_1 + n_2)$ equations as defined by Equation (1). This gives us $T \times (n_1 + n_2)$ equations;

- for $t = 2, \ldots, T$, rational expectations requires $\eta_t = 0$. From the perspective of period $t = 1$, there should be no forecast errors or revisions to expectations. This gives us $(T-1) \times k$ equations; and

- if a stable solution exists for $t = T+1$ onwards, then $\bar{Z}_2' y_T = \bar{w}_{2,T}$, where $\bar{w}_{2,T}$ is given by

$$\bar{w}_{2,T} = \left(\bar{\Lambda}_{22} - \bar{\Omega}_{22} \right)^{-1} \bar{Q}_2 \bar{C} - \mathbb{E}_1 \sum_{j=1}^{\infty} (\bar{\Omega}_{22}^{-1} \bar{\Lambda}_{22})^{j-1} \bar{\Omega}_{22}^{-1} \bar{Q}_2 \Psi \varepsilon_{T+j} \qquad (19)$$

Equation (19) gives $\bar{m}$ equations where $\bar{m}$ represents the number of explosive eigenvalues of the final (bar) system. $\bar{Z}_2'$ is from the QZ decomposition of $(\bar{\Gamma}_0, \bar{\Gamma}_1)$ and therefore has $\bar{m}$ independent rows. The last condition is effectively a terminal condition that guarantees that the system is on its SSP for $t \geq T+1$.

In total, we get $T \times (n_1 + n_2 + k) + \bar{m} - k$ equations that can be summarised as follows

$$
\begin{pmatrix}
\tilde{\Gamma}_{0,1} & 0 & \cdots & \cdots & 0 \\
-\Gamma_{1,2} & \Gamma_{0,2} & \ddots & & \vdots \\
0 & -\Gamma_{1,3} & \Gamma_{0,3} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -\Gamma_{1,T} & \Gamma_{0,T} \\
0 & \cdots & \cdots & 0 & \bar{Z}_2'
\end{pmatrix}
\begin{pmatrix}
y_1 \\
\vdots \\
y_T
\end{pmatrix}
=
\begin{pmatrix}
\tilde{C}_1 + \tilde{\Gamma}_{1,1} y_0 + \tilde{\Psi}_1 \varepsilon_1^a \\
C_2 + \Psi_2 \varepsilon_2^a \\
\vdots \\
C_T + \Psi_T \varepsilon_T^a \\
\bar{w}_{2,T}
\end{pmatrix}
\tag{20}
$$

The condition that $\eta_2, \ldots, \eta_T = 0$ implies that $\Pi \eta_t = 0$ for $t = 2, \ldots, T$. Also notice that the structure of $\{\Gamma_{0,t}, \Gamma_{1,t}, C_t, \Psi_t\}_{t=2}^T$ guarantees that $\mathbb{E}_t z_{t+1} = z_{t+1}$ since the last k rows of C_t and Ψ_t are zero for all t.

Solving for $y_1, \ldots, y_T$ involves solving a linear system of the form $Ay = b$, where $y = (y_1', \ldots, y_T')'$ and A stands for the matrix on the left while b stands for the vector on the right hand side of Equation (20). Propositions 1 and 2 imply that for the final (bar) system to have a unique solution, $\bar{m} = k$; in this case, Equation (20) has as many equations as there are unknowns. However, if the final bar system has many solutions, $\bar{m} < k$, then Equation (20) forms a system with less equations than unknowns, in which case, if there is a solution, there are infinitely many. Obviously, the existence of a solution to the structurally invariant final system is a necessary condition for Equation (20) to have a solution. We summarise these two observations with the following propositions.

Proposition 3. *Existence of a solution to the final bar system*

$$
\bar{\Gamma}_0 y_t = \bar{C} + \bar{\Gamma}_1 y_{t-1} + \bar{\Psi} \varepsilon_t + \bar{\Pi} \eta_t
$$

is necessary for the existence of a solution to Equation (20).

Proposition 4. *Uniqueness of a solution to the final bar system*

$$
\bar{\Gamma}_0 y_t = \bar{C} + \bar{\Gamma}_1 y_{t-1} + \bar{\Psi} \varepsilon_t + \bar{\Pi} \eta_t
$$

is necessary for the uniqueness of a solution to Equation (20).

The propositions above state necessary but not sufficient conditions for the existence and uniqueness of a solution for $\{y_t\}_{t=1}^{T}$. The existence and uniqueness of a solution for $\{y_t\}_{t=1}^{T}$ ultimately depend on the properties of the matrix A. We have shown that if a unique solution exists for the final structure, A is a square matrix. Next, we argue that A will generally be a full-rank matrix for the following reasons:

- The rank of the matrix $\begin{pmatrix} -\Gamma_{1,t} & \Gamma_{0,t} \end{pmatrix}$ is n for $t = 2, \ldots, T$. If not, there are linearly dependent and possibly inconsistent equations; a sort of ill-specified problem.

- The block bi-diagonal structure of the matrix A implies that none of the rows associated with period t can be obtained as a linear combination of rows associated with non-adjacent periods. If this is the case, this implies that for some period, the rank of the matrix $\begin{pmatrix} -\Gamma_{1,t} & \Gamma_{0,t} \end{pmatrix}$ will be less than n for some t, violating the preceding point.

- For a well-defined system, the rows of $\tilde{\Gamma}_{0,1}$ will be linearly independent. So the first $n_1 + n_2$ rows of A will be linearly independent.

- The rows of $\bar{Z}_2'$ are linearly independent because $\bar{Z}$ is unitary. So the last k rows of A will be linearly independent.

- In general, no row, for a given period, can be expressed as a linear combination of the rows associated with that period and from an adjacent period. This would mean that $\Gamma_{1,t}$ and $\Gamma_{0,t+1}$ are rank deficient. Even if this were the case, suppose for non-zero vectors, w and v, $w\Gamma_{1,t} = v\Gamma_{0,t+1} = 0$, then we would also require that $w\Gamma_{0,t} - v\Gamma_{1,t+1} = 0$ if there were a linear dependency in rows associated with periods t and $t+1$. Although this is possible, we argue that this seems unlikely.

- The last k rows are linearly independent of the first $nT - k$ rows. Clearly, the last k rows are linearly independent of the rows associated with periods $1, \ldots, T-1$, for the same reasons we discussed earlier for non-adjacent periods. But, in general, the last k rows of A are linearly independent of the preceding n rows. If a linear combination of the rows of $\bar{Z}_2'$ reproduce a row of $\Gamma_{0,T}$, that same linear combination of zero vectors must reproduce the corresponding row of $-\Gamma_{1,T}$, which may not necessarily be zero. $\bar{Z}_2'$ is typically unrelated to $\Gamma_{0,T}$ because it comes from the QZ decomposition of $(\bar{\Gamma}_0, \bar{\Gamma}_1)$. But even if $\bar{Z}_2'$ came from the QZ decomposition of $(\Gamma_{0,T}, \Gamma_{1,T})$, it relates to $\Gamma_{0,T}$ in a non-linear fashion.

The arguments above imply that the matrix *A* will be invertible, in which case, the solution for $\{y_t\}_{t=1}^{T}$ will be unique. However, it is possible that *A* is not invertible under some perverse parameter variations. Under such circumstances, the existence of solution requires *b* to be contained in the column space of the matrix *A*; but this is not a guarantee that any such solution will be unique. The invertibility of *A* obviously guarantees that there is a unique solution.

This suggests that the way in which parameters vary can determine whether a unique solution exists or not. For instance, should a policy maker decide to change the parameters of the policy rule over a set length of time, it might matter how this policy change is implemented over time for a unique equilibrium path to exist.

So we conclude that, in general, the existence and uniqueness of a solution for $\{y_t\}_{t=1}^{T}$ will hinge on the existence and uniqueness of a solution to the final structure.

The solution method we propose has a number of advantages: it is simple to implement as it only requires solving a matrix inversion problem; even in the absence of structural changes, it enables us to forecast over finite horizons without resorting to loops; and it can be used recursively to produce stochastic simulations in the face of fully predictable structural variations.

4.   Numerical Examples

4.1   The Model

We illustrate the solution method outlined above with a series of examples using a version of the New-Keynesian model presented in Ireland (2007). Unlike Ireland (2007), we assume, for simplicity, that the gross inflation target is a policy-determined constant, that the deviation of the technology process from its steady state, $\hat{z}_t$, follows a stationary process, and that there are no habits in consumption. Under these assumptions it is easy to show that the equilibrium obeys the following set of log-linear equations

$$\hat{y}_t = -\sigma^{-1}(r_t - E_t\pi_{t+1}) + E_t\hat{y}_{t+1} + \frac{(1-\rho_a)}{\sigma}\hat{a}_t - \frac{1}{\sigma}\ln\beta \tag{21}$$

$$\pi_t = \frac{1}{(1+\beta\alpha)}\bigg((1+\beta\alpha-\alpha-\beta)\pi^* + \alpha\pi_{t-1} + \psi\sigma\hat{y}_t$$

$$-\psi\hat{z}_t + \beta E_t\pi_{t+1} - \hat{e}_t\bigg) \tag{22}$$

$$r_t = (1-\rho_r)r + \rho_r r_{t-1} + \rho_\pi(\pi_t - \pi^*) + \rho_y\hat{y}_t + \rho_g\hat{g}_t + \varepsilon_{r,t} \tag{23}$$

$$\hat{g}_t = \hat{y}_t - \hat{y}_{t-1} \tag{24}$$

$$\hat{a}_t = \rho_a\hat{a}_{t-1} + \varepsilon_{a,t} \tag{25}$$

$$\hat{e}_t = \rho_e\hat{e}_{t-1} + \varepsilon_{e,t} \tag{26}$$

$$\hat{z}_t = \rho_z\hat{z}_{t-1} + \varepsilon_{z,t} \tag{27}$$

Equations (21),(22), (23), and (24) are the 'IS-curve', Phillips curve, Taylor-rule, and definition of the growth rate of output, while (25),(26), and (27) govern the behaviour of the exogenous shock processes to demand, $\hat{a}_t$, the mark-up, $\hat{e}_t$, and technology, $\hat{z}_t$. In the equations above, π_t is the log gross rate of inflation between periods $t-1$ and t; π^* stands for the log of the target rate of inflation; $\hat{y}_t = \ln(Y_t/Y)$ is the percentage deviation of output from its steady state level, Y; $\hat{g}_t$ is the growth rate of output; $r_t = \ln R_t$ stands for the log of the gross nominal interest rate between periods t and $t+1$.; $r = \pi^* - \ln\beta$ is the steady state level of r_t; β is the household's discount factor; σ is the inverse of the intertemporal elasticity of substitution; $\alpha \in [0,1]$ governs the degree to which price-setting is 'backward-looking'; the parameters ρ_a, ρ_e, and ρ_z all $\in [0,1)$; and $\psi = (\theta-1)/\phi$ is defined for convenience, where θ is the steady state elasticity of substitution between intermediate goods and ϕ controls the magnitude of price adjustment costs. Finally, $\varepsilon_{a,t}$, $\varepsilon_{e,t}$, $\varepsilon_{r,t}$ and $\varepsilon_{z,t}$ are all assumed to be independent and identically distributed (iid) disturbances with mean zero and standard deviations σ_a, σ_e, σ_r, and σ_z respectively.

While some variables are expressed in percentage deviations from their steady state values, others, like π_t and r_t, are left expressed in log-levels. The only reason for this is that it aids in the interpretation of the numerical examples that follow – in particular, those that involve changes in the steady state values of these same

variables. For example, a change of the inflation target alters the steady state values of inflation and the nominal interest rate.

The set of equations given by (21), (22), (23), (25), (26), and (27), together with the definitions for the one-period ahead forecast errors can be easily put in the form $\Gamma_0 y_t = C + \Gamma_1 y_{t-1} + \Psi \varepsilon_t + \Pi \eta_t$.

We set the model's parameters to obtain a benchmark calibration for the numerical examples that follow. The parametrisation (Table 1) is inspired empirically and in cases borrows values from the literature of similarly estimated models. The parametrisation itself is, for our purposes unimportant.

<div align="center">

Table 1: Parameter Values

</div>

| Parameter	Description	Value
$\pi^\star$	Inflation target	0.0125
β	Household's discount factor	0.9925
$\frac{1}{\sigma}$	Intertemporal elasticity of substitution	1.0
α	Backward-looking price-setting	0.25
ψ	Elasticity of substitution adjusted for price adjustment costs	0.1
ρ_r	Persistence of nominal interest rate	0.65
ρ_π	Policy rule inflation coefficient	0.5
ρ_y	Policy rule output gap coefficient	0.1
ρ_g	Policy rule output growth coefficient	0.2
ρ_a	Persistence of demand shocks	0.9
σ_a	Standard deviation of demand disturbance	0.02
ρ_z	Persistence of technology shocks	0.9
σ_z	Standard deviation of technology disturbance	0.007
ρ_e	Persistence of mark-up shocks	0.9
σ_e	Standard deviation of mark-up disturbance	0.001
σ_r	Standard deviation of monetary disturbance	0.002

4.2 An Increase in ρ_π

We start by considering the impact of announcing a more aggressive policy towards inflation. The announcement refers to the future value of ρ_π. In particular, the two structures differ only with respect to their value of ρ_π. Both structures, however, share the same steady state. The initial structure is that of the benchmark parametrisation which sets ρ_π to 0.5. The final structure sets ρ_π to 1.

The announcement of a different future value for ρ_π has no effect on the evolution of the non-stochastic steady state; the dynamics are uninteresting if the system begins and remains at its steady state. To study the implications of an anticipated structural change we consider a persistent demand shock and assume that the economy is away from its steady state when this information is known.

Figure 1 shows impulse responses of output, inflation, the nominal interest rate, and output growth to a one standard deviation demand shock, $\varepsilon_{a,t}$. The dashed blue lines show conventional impulse responses given the initial structure: the impulse responses that would have prevailed in the absence of any known future change in policy. Similarly, the dashed red lines show the conventional impulse responses under the final structure: the responses that would have prevailed had the new rule always been in place. The green solid lines show the equilibrium responses of a credible announcement made in period 4 that in period 8 a new policy that sets ρ_π equal to 1.0 would be in place.

Figure 1: IRF to a demand shock with an anticipated future change of ρ_π

Announcement in 4, implementation in 8
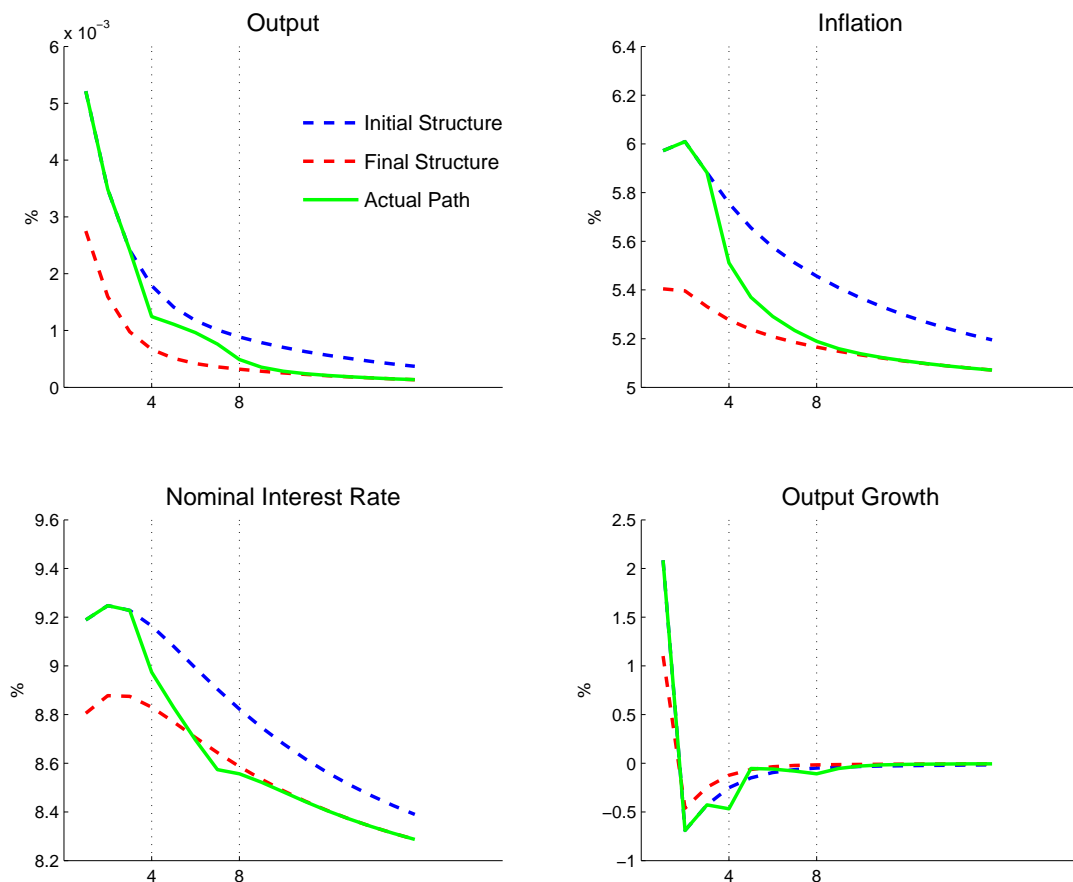
For the first three periods, the economy behaves according to the initial structure. At the time of the announcement, inflation, output and the interest rate jump to their new stable saddle path. Because of forward-looking price-setting behaviour, the credible announcement of a more aggressive policy towards inflation in the future serves to reduce inflation relative to the response in the absence of any announcements. In period 4, annualised inflation would have been around 5.7 per cent, but the announcement has the effect of bringing inflation down to 5.5 per cent. In period 7, prior to the actual implementation of the new policy rule, inflation is already at 5.2 per cent, close to where inflation would have been had the final structure always been in place.

The way the economy evolves between the announcement and the implementation is essentially a function of the length of the intervening period and also of the 'distance', so to speak, between the initial and final reduced forms. If, for example, the announcement involves a change which is far into the future, then its contemporaneous impact would be small. In fact, for the first few periods the economy's response would be fairly similar to those of the prevailing structure. This is illustrated in Figure 2 which compares the response of inflation in Figure 1 with the one that would prevail if the new policy is implemented in period 24 (instead of period 8).

The more the announcement alters the future reduced form of the system, relative to its present one, the more strongly the economy reacts in the intervening period.

4.3   A Change of the Inflation Target

We consider the impact of announcing a lower inflation target. The announcement refers to the future value of π^*. As before, we assume that a demand shock hits the economy in period 1 and that at the beginning of period 4 the central bank announces that it will implement a lower inflation target from period 8 onwards. In this case, notice that the two structures differ only with respect to the steady state value of their nominal quantities. The impulse response functions for output are invariant to the level of the inflation target (Figure 3). However, because of the presence of nominal rigidities, the announcement has real effects.

As Figure 3 shows, both inflation and output fall after the announcement is made. The central bank conducts policy as governed by the initial policy, however, which implies a departure of inflation well below its initial inflation target. Although

Figure 2: **Length of Intervening Period: IRF of inflation to a demand shock with an anticipated future change of ρ_π**

Announcement in 4, implementation in 8 versus 24
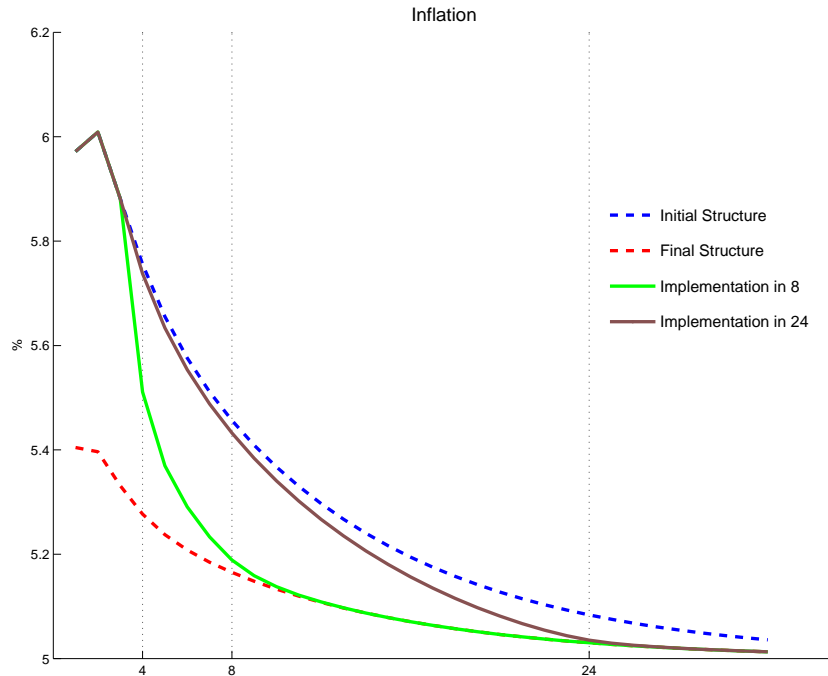
the central bank cuts the nominal interest rate, the real interest rate increases and output growth consequently falls.

4.4 Announced Shocks

Figure 4 shows the responses of inflation, output and the nominal interest rate to an announced sequence of monetary policy shocks (as shown in the bottom right panel). In period 2, the monetary authority announces that a sequence of deviations from the prevailing rule will occur. In particular, these are expansionary shocks to the policy rule and consequently inflation and output increase in the announcement period. Inflation and the nominal interest rate reach their peaks in period 3 before the announced shocks take place. Unlike the case with unanticipated shocks, as these fully anticipated shocks occur, output, the nominal interest rate and inflation are already gradually returning towards their steady state values.

Figure 3: **IRF to a demand shock with an anticipated change of π^* from 5 to 2.5 per cent per annum**

Announcement in 4, implementation in 8
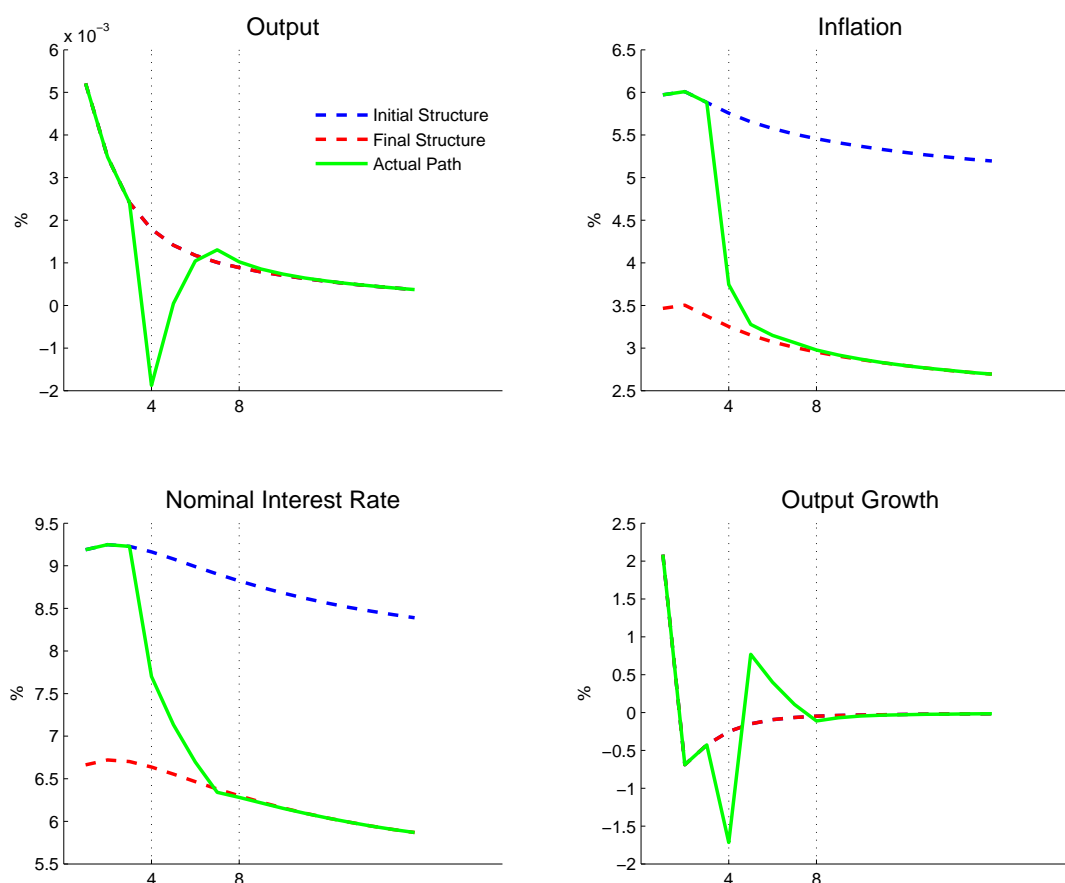
4.5   A Stochastic Simulation: Announcing a New Monetary Policy Rule

As we have discussed above, the solution method described in Section 3 can be used recursively to conduct stochastic simulations. The green lines in Figure 5 show a stochastic simulation with the following characteristics: the benchmark parametrisation with an inflation target of 15 per cent per annum until period 50; at which time it is announced that in 12 periods time, a lower inflation target of 2 per cent per annum and a more aggressive long-run response to inflation deviations from that the new target will be in place: $\bar{\rho}_\pi = \bar{\rho}_r = 1$. The different policy rules give rise to different dynamics, as one would expect. It is interesting to note, however, that the properties of the new regime seem to be inherited shortly after the announcement is made.

Figure 4: Response to an announced sequence of policy shocks
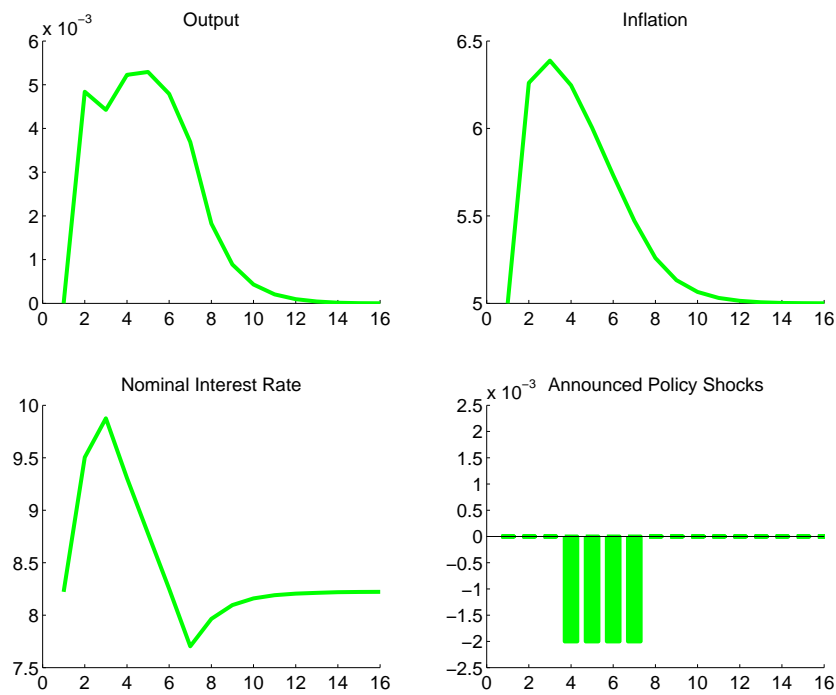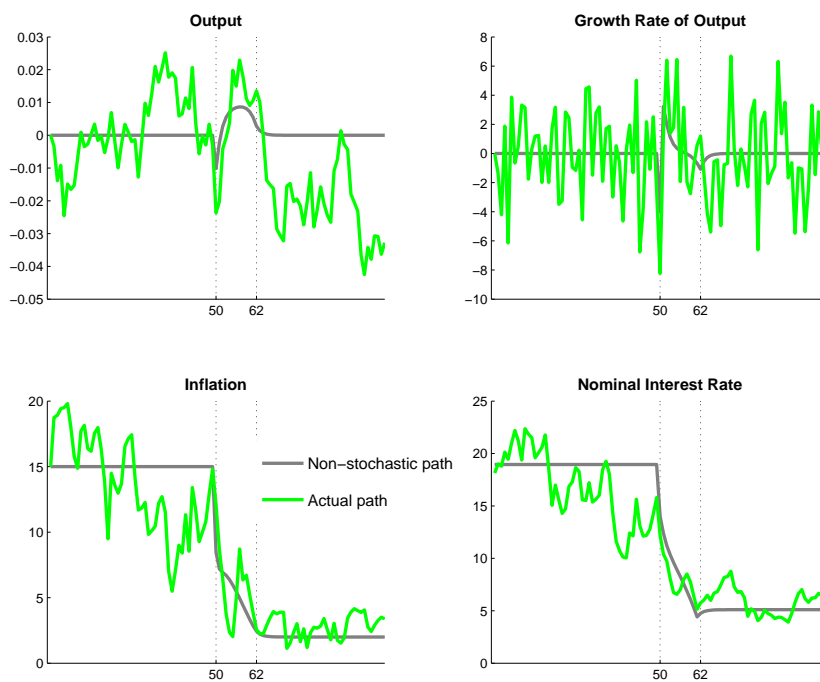### Announcement made in period 2

Figure 5: A Stochastic Simulation: Change in the inflation target from 15 per cent to 2 per cent, and in ρ_π to 1 and ρ_r to 1
Announcement in period 50, implementation in period 62

5.   Conclusions

We have outlined a technique to solve linear rational expectations models in the face of anticipated changes to the parameters or exogenous variables. This solution has a number of important applications. Pre-announced changes to a policy rule can be examined using the techniques discussed in this paper. Variations in the response of monetary policy to the state of the economy, adjustments to the monetary policy objectives or anticipated deviations from a policy rule can all be analysed using the methods outlined in this paper. In more fully specified models, one can examine the consequences of shifting from one policy regime (such as monetary targeting) to another policy regime (inflation targeting). Of course, the method we propose also deals with other anticipated changes to the structure of an economy.

We have shown that the properties of the final structure are crucial for the way the system behaves between the time agents become aware of a forthcoming event and the time that the event occurs.

The results have implications to the application of policy rules. For example, if a policy maker uses a monetary policy rule that does not satisfy the Taylor principle, then a unique rational expectations equilibrium typically does not exist. These rules are considered "bad" as they lead to multiple equilibria. However, if the policy maker makes a credible announcement that they will adopt a better rule, one that satisfies the Taylor principle, at some point in the future, then a unique equilibrium will exist for the economy – regardless of exactly when this will occur or how bad the policy rule is in the intervening period.

We have assumed that all announcements, for instance of an impending policy change, are credible. Further research could extend these techniques to examine the effect of such announcements when credibility is less than perfect.

Appendix A: Proof of Propositions 1 and 2

Proof of Proposition 1:

Proof.
Sufficiency: If $rank(Q_2\Pi) = m$, then the columns of $Q_2\Pi$ span $\mathbb{R}^m$. This means that for arbitrary initial conditions and for any fundamental shock, expectation revisions can keep the system on its SSP.

Necessity: $w_{2,t}$ must satisfy:

$$\Lambda_{22}w_{2,t} = \Omega_{22}w_{2,t-1} + Q_2(C + \Psi\varepsilon_t + \Pi\eta_t) \tag{A1}$$

To be on the SSP, $w_{2,t}$ must also satisfy

$$
\begin{aligned}
w_{2,t} &= (\Lambda_{22} - \Omega_{22})^{-1}Q_2C - \sum_{j=1}^{\infty}(\Omega_{22}^{-1}\Lambda_{22})^{j-1}\Omega_{22}^{-1}Q_2(\Psi\varepsilon_{t+j} + \Pi\eta_{t+j}) \\
&= (\Lambda_{22} - \Omega_{22})^{-1}Q_2C - \mathbb{E}_t\sum_{j=1}^{\infty}(\Omega_{22}^{-1}\Lambda_{22})^{j-1}\Omega_{22}^{-1}Q_2\Psi\varepsilon_{t+j}
\end{aligned}
$$

and more specifically

$$w_{2,1} = (\Lambda_{22} - \Omega_{22})^{-1}Q_2C - \mathbb{E}_1\sum_{j=1}^{\infty}(\Omega_{22}^{-1}\Lambda_{22})^{j-1}\Omega_{22}^{-1}Q_2\Psi\varepsilon_{j+1}$$

Now suppose the initial condition of the system, y_0, is arbitrary, so that the economy may not necessarily be on the SSP. We can then write $w_{2,0} = Z_2'y_0$ as the sum of a component that is on the SSP and some deviation from the SSP value

$$w_{2,0} = (\Lambda_{22} - \Omega_{22})^{-1}Q_2C - \mathbb{E}_0\sum_{j=1}^{\infty}(\Omega_{22}^{-1}\Lambda_{22})^{j-1}\Omega_{22}^{-1}Q_2\Psi\varepsilon_j + \Delta_0$$

where $\Delta_0 \in \mathbb{R}^m$. We can look at Equation (A1) from the perspective of period 1 to show that in order for equality to hold (that is, for the system to be on the stable

saddle path in period 1), the following condition must be satisfied

$$Q_2\Pi\eta_1 = -\Omega_{22}\Delta_0 + \Omega_{22}\sum_{j=1}^{\infty}(\Omega_{22}^{-1}\Lambda_{22})^{j-1}\Omega_{22}^{-1}Q_2\Psi(\mathbb{E}_0\varepsilon_j - \mathbb{E}_1\varepsilon_j)$$

since Δ_0 is arbitrary, we require the columns of $Q_2\Pi$ to span $\mathbb{R}^m$. Since $Q_2\Pi$ is $m \times k$, this is equivalent to requiring $rank(Q_2\Pi) = m$.

Proof of Proposition 2:

Proof.
Sufficiency: Suppose that $rank(Q_2\Pi) = k$, then the rows of $Q_2\Pi$ span $\mathbb{R}^k$. Therefore, $rowspace(Q_1\Pi) \subseteq rowspace(Q_2\Pi)$, since the rows of $Q_1\Pi$ necessarily span some subspace of $\mathbb{R}^k$.

Necessity: Suppose that the solution is unique. This means that $rowspace(Q_1\Pi) \subseteq rowspace(Q_2\Pi)$.

We know that Q is a full rank $n \times n$ matrix. Post multiplying by Π extracts the last k columns of Q. Since Q was full rank, $Q\Pi$ must have rank k. If the solution is unique, then this means that the rank of

$$\begin{pmatrix} Q_1\Pi \\ Q_2\Pi \end{pmatrix}$$

should have the same rank as that of $Q_2\Pi$. But since the matrix above is exactly $Q\Pi$, this implies that the rank of $Q_2\Pi$ is k.

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