The Unit-effect Normalisation in Set-identified Structural Vector Autoregressions

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Abstract

Structural vector autoregressions that are set-identified (e.g. using sign restrictions) are typically estimated under the normalisation that the structural shocks have unit standard deviation, in which case the estimated impulse responses are to a standard-deviation shock. However, impulse responses to a unit shock – a shock that raises a particular variable by one unit – are often of greater relevance, particularly for policy analysis. For example, central bankers are interested in answering questions like 'what are the effects of a 100 basis point increase in the federal funds rate?' This paper explores the extent to which set-identifying restrictions are informative about impulse responses to unit shocks. I show that identified sets for these impulse responses may be unbounded and discuss issues that this raises for conducting inference. I explain how to draw useful posterior inferences about impulse responses even when the identified sets for these impulse responses are unbounded at some values of the reduced-form parameters. I illustrate the empirical relevance of these issues by estimating the macroeconomic effects of a 100 basis point shock to the federal funds rate. The results obtained under a rich set of sign and narrative restrictions are broadly consistent with the effects of US monetary policy on output lying at the smaller end of the range of existing estimates.

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1. Introduction

When estimating the effects of macroeconomic shocks using structural vector autoregressions (SVARs), it has become increasingly common to use sign restrictions and/or a set of zero restrictions that are insufficient to point-identify the parameters of interest, in which case the parameters are set-identified (e.g. Uhlig 2005; Arias, Rubio-Ramírez and Waggoner 2018).[1] Set-identified SVARs are typically estimated under the normalisation that the structural shocks have unit standard deviation (i.e. the 'standard-deviation normalisation'). The impulse responses that are obtained under this normalisation consequently represent impulse responses to a standard-deviation shock. However, as argued by Fry and Pagan (2011) and Stock and Watson (2016, 2018), impulse responses to a unit shock – a shock that raises a particular variable by one unit – are naturally more relevant for policy analysis. For example, central bankers are interested in answering questions like 'what are the effects of a 100 basis point increase in the policy rate?'

In this paper, I explore the extent to which set-identifying restrictions are informative about impulse responses to unit shocks (i.e. under the 'unit-effect normalisation'). In particular, I show that the 'identified set' for an impulse response to a unit shock – the set of values of the impulse response that are consistent with the reduced-form parameters given the identifying restrictions – may be unbounded. To give some intuition, the impulse responses to a unit shock are obtained by dividing the impulse response of a particular variable with respect to a standard-deviation shock by the 'normalising impulse response' (e.g. the impact response of the federal funds rate with respect to a standard-deviation monetary policy shock). When the identified set for the normalising impulse response includes zero, it may be possible to make the impulse response to a unit shock arbitrarily large by considering a sequence of parameters converging to the point where the normalising impulse response is zero. The possibility that the identified set for the impulse response to a unit shock is unbounded suggests that set-identifying restrictions have the potential to be extremely uninformative about these impulse responses. I demonstrate the implications of an unbounded identified set when conducting inference, with a focus on the prior-robust approach to Bayesian inference proposed in Giacomini and Kitagawa (2021), and discuss how researchers can potentially draw useful posterior inferences about impulse responses to a unit shock when the identified set is unbounded with positive posterior probability.[2]

Under the standard approach to Bayesian inference in set-identified SVARs (e.g. Uhlig 2005; Rubío-Ramirez, Waggoner and Zha 2010; Arias *et al* 2018), it is straightforward to transform from the standard-deviation normalisation to the unit-effect normalisation. As is the case under point-identifying restrictions, this transformation simply requires dividing the impulse responses obtained under the standard-deviation normalisation by the normalising impulse response. Repeating this at each draw of the parameters from their posterior distribution generates a posterior distribution for the impulse responses to a unit shock. However, there are well-documented problems with the standard approach to Bayesian inference in set-identified models. In particular, because the model

1 Loosely speaking, a parameter is point-identified if its value can be pinned down given knowledge of the joint distribution of the data. A parameter is set-identified if its value can only be determined up to a (non-singleton) set.

2 When applying their robust Bayesian approach to inference to a set-identified SVAR, Giacomini and Kitagawa (2021) focus on the impulse responses to standard-deviation shocks as the parameters of interest. Giacomini, Kitagawa and Read (2022) describe an algorithm for conducting robust Bayesian inference in proxy SVARs (i.e. SVARs identified using an external instrument) under the unit-effect normalisation. They note that the identified set may be unbounded at some values of the reduced-form parameters, but do not draw out the implications of this issue for conducting inference.

is set-identified, the likelihood function is flat with respect to certain parameters. As a consequence, a component of the prior is 'unrevisable' in the sense that it is never updated and, as a consequence, posterior inference may be sensitive to the choice of prior (Poirier 1998; Baumeister and Hamilton 2015).[3]

To address the problem of posterior sensitivity, Giacomini and Kitagawa (2021) propose conducting Bayesian inference in set-identified models using an approach that is robust to the choice for the unrevisable component of the prior (a 'robust Bayesian approach' to inference). The key feature of this approach is that it replaces the prior with a *class* of priors, which contains all priors that are consistent with the identifying restrictions (given a prior for the reduced-form parameters).[4] The class of priors generates a class of posteriors, which can be summarised in various ways. For example, rather than generating a single posterior mean, the class of posteriors generates a *set* of posterior means, which is an interval that contains every posterior mean that could be obtained under the class of priors. The class of posteriors can also be summarised using a 'robust credible interval', which is an interval that is assigned at least a given posterior probability under all posteriors in the class. Additionally, the class of posteriors generates a set of posterior probabilities for any particular hypothesis of interest (e.g. that the output response is negative at some horizon). This set can be summarised by the posterior lower and upper probabilities, which are, respectively, the smallest and largest posterior probabilities of the hypothesis over the class of posteriors.

In the context of set-identified SVARs, implementing the robust Bayesian approach to inference requires computing the lower and upper bounds of the identified set for each impulse response. As noted above, if zero is contained within the identified set for the normalising impulse response, the identified sets for the impulse responses to a unit shock may be unbounded. In turn, if these identified sets are unbounded within any region of the reduced-form parameter space that receives positive posterior probability, the sets of posterior means will be unbounded. At face value, this suggests that set-identifying restrictions may be extremely uninformative about the impulse responses to a unit shock.[5] Nevertheless, it may be possible to draw useful posterior inferences about impulse responses to a unit shock when the identified set is unbounded with positive posterior probability. For example, even when the set of posterior *means* is unbounded, the set of posterior *medians* or some other quantile may be bounded, and it may be possible to construct robust credible intervals if the credibility level is not too extreme. Moreover, the posterior lower and upper probabilities remain well defined. Consequently, it is always possible to draw inferences such as 'the posterior probability that output declines by more than x per cent at horizon h in response to a 100 basis point monetary policy shock is at least y per cent and at most z per cent'.

Given the ubiquity of Bayesian methods in the literature on set-identified SVARs, and the well-documented problem of posterior sensitivity to the choice of prior in these models, I focus on the implications of unbounded identified sets for conducting robust Bayesian inference. However,

3 The posterior density is the product of the likelihood and the prior density. Conditional on the reduced-form parameters, the likelihood function is flat, so the posterior will be proportional to the prior.

4 An alternative approach to Bayesian inference in SVARs is to impose a prior directly over the structural parameters (as advocated in Baumeister and Hamilton (2015, 2018, 2019)). It remains the case under this approach that a component of the prior will never be updated by the data, so posterior sensitivity to the choice of prior may still be a concern.

5 So long as the VAR is stable, the identified set for the impulse response to a standard-deviation shock is always bounded. The issue of unboundedness is therefore specific to the case where the impulse response of interest is with respect to a unit shock.

unboundedness may also arise when estimating or conducting inference about impulse responses to a unit shock within a frequentist framework. Existing approaches to frequentist inference in set-identified SVARs focus on impulse responses to a standard-deviation shock as the parameters of interest (e.g. Gafarov, Meier and Montiel Olea 2018; Granziera, Moon and Schorfheide 2018). If the maximum-likelihood estimator (MLE) of the reduced-form parameters is such that zero is included within the identified set for the normalising impulse response, a frequentist estimate of the identified set for impulse responses to a unit shock may be unbounded.

To make these issues clear, I use a bivariate SVAR in which I can analytically characterise identified sets under some sign restrictions on impulse responses. I then explain how to verify whether identified sets for the impulse responses to a unit shock may be unbounded in an SVAR of arbitrary dimension identified using both sign and zero restrictions. I first show that a *necessary* condition for unboundedness of these identified sets is that the identified set for the normalising impulse response includes zero. I explain why this condition is not also sufficient for unboundedness by describing an example where the condition is satisfied but particular impulse responses to unit shocks are bounded. I then provide an easily verifiable *sufficient* condition under which the identified set for the normalising impulse response includes zero; specifically, if the number of sign and zero restrictions is less than the dimension of the SVAR and the restrictions relate to a single structural shock, the identified set for the normalising impulse response always includes zero. When this sufficient condition is not satisfied (i.e. when there are more restrictions than variables in the SVAR and/or the restrictions relate to multiple shocks), I explain how to check whether the identified set for the normalising impulse response includes zero at a given value of the reduced-form parameters by adapting numerical algorithms that have been previously used to check whether identified sets are nonempty (e.g. Giacomini and Kitagawa 2021; Read, forthcoming).

To illustrate the importance of these issues in practice, I estimate the macroeconomic effects of a 100 basis point shock to the federal funds rate under different combinations of identifying restrictions: the sign restrictions on impulse responses to a monetary policy shock proposed in Uhlig (2005); the sign and zero restrictions on the systematic component of monetary policy proposed in Arias, Caldara and Rubio-Ramírez (2019); and the 'narrative restrictions' proposed in Antolín-Díaz and Rubio-Ramírez (2018).

Under the restrictions considered in Arias *et al* (2019), the sufficient condition described above is satisfied, so zero is always included in the identified set for the normalising impulse response.[6] This suggests that identified sets for the impulse responses to a 100 basis point shock may always be unbounded. Numerical approximations of the bounds of the identified set suggest that this is indeed the case. These restrictions are therefore extremely uninformative about the effects of a 100 basis point shock, and outputs obtained using standard Bayesian inference are misleading about the informativeness of the data and identifying restrictions. Combining these restrictions with the sign restrictions on impulse responses considered in Uhlig (2005) yields identified sets that are bounded with posterior probability close to, but less than, 100 per cent. In this case, since the identified sets appear to be unbounded with positive posterior probability, the sets of posterior means for the output response to a 100 basis point shock are unbounded. Nevertheless, the sets of posterior *medians* remain bounded, because the identified sets are unbounded with low posterior probability.

6 I do not make a judgement about whether it is reasonable that the federal funds rate does not respond on impact to a monetary policy shock. I seek to clarify that existing identifying restrictions do not necessarily rule this possibility out, and the implications that this has for drawing inferences about the effects of unit shocks.

The set of posterior medians for the output response includes zero at almost all horizons of interest, and the restrictions are unable to rule out large *increases* in output following a positive 100 basis point shock.

Additionally imposing narrative restrictions on the sign of the monetary policy shock in October 1979 and its contribution to the change in the federal funds rate in this month (as in Antolín-Díaz and Rubio-Ramírez (2018)) results in the identified sets being bounded with 100 per cent posterior probability. This implies that the sets of posterior means and all posterior quantiles are bounded. The additional restrictions substantially tighten the set of posterior medians and robust credible intervals. The results under this set of restrictions are consistent with the largest effects of monetary policy on output occurring after about two years and lying towards the lower end of the range of existing estimates summarised in Ramey (2016).

Finally, I discuss the possibility of using alternative identifying restrictions to ensure that the identified sets for the impulse responses to a unit shock are bounded. A straightforward solution would be to directly bound the normalising impulse response away from zero; however, I argue that it may be difficult to justify such restrictions and inferences may be extremely sensitive to changes in the imposed bound. A lower bound on the forecast error variance decomposition, such as those proposed in Volpicella (forthcoming), will indirectly constrain the impulse response of the normalising variable to be nonzero, but – again – such restrictions may be difficult to justify in practice and yield results that are highly sensitive to the imposed bound.

The remainder of the paper is structured as follows. Section 2 outlines the SVAR framework and the robust Bayesian approach to inference. Section 3 uses a bivariate SVAR to outline the issues associated with conducting inference about impulse responses to a unit shock. Section 4 describes how to check whether identified sets for impulse responses to a unit shock are unbounded in a more general setting. Section 5 estimates the macroeconomic effects of a 100 basis point shock to the federal funds rate under different sets of identifying restrictions. Section 6 discusses using alternative restrictions to ensure boundedness of the identified sets. Section 7 concludes. Proofs and additional details are contained in the appendices.

Notation. For a matrix $\boldsymbol{X}$, $\mathrm{vec}(\boldsymbol{X})$ is the vectorisation of $\boldsymbol{X}$. When $\boldsymbol{X}$ is symmetric, $\mathrm{vech}(\boldsymbol{X})$ is the half-vectorisation of $\boldsymbol{X}$, which stacks the elements of $\boldsymbol{X}$ that lie on or below the diagonal into a vector. $\boldsymbol{e}_{i,n}$ is the ith column of the $n \times n$ identity matrix, $\boldsymbol{I}_n$. $\mathbf{0}_{n\times m}$ is an $n \times m$ matrix of zeros.

2. Framework

This section describes the SVAR model, outlines the concepts of identifying restrictions and identified sets, and describes the robust Bayesian approach to inference.

2.1 SVAR and Orthogonal Reduced Form

Let $\boldsymbol{y}_t$ be an $n \times 1$ vector of variables following the SVAR(p) process:

$$\boldsymbol{A}_0\boldsymbol{y}_t = \boldsymbol{A}_+\boldsymbol{x}_t + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon_t} \sim N(\mathbf{0}_{n\times 1}, \boldsymbol{I}_n),$$

where $\boldsymbol{A}_0$ is an invertible $n \times n$ matrix with positive diagonal elements (which is a normalisation on the signs of the structural shocks) and $\boldsymbol{x}_t = (\boldsymbol{y}'_{t-1}, \dots, \boldsymbol{y}'_{t-p})'$. The 'orthogonal reduced form' of the model is:

$$\boldsymbol{y}_t = \boldsymbol{B}\boldsymbol{x}_t + \boldsymbol{\Sigma}_{tr}\boldsymbol{Q}\boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{B} = (\boldsymbol{B}_1, \dots, \boldsymbol{B}_p) = \boldsymbol{A}_0^{-1}\boldsymbol{A}_+$ is the matrix of reduced-form coefficients, $\boldsymbol{\Sigma}_{tr}$ is the lower-triangular Cholesky factor of the variance-covariance matrix of the reduced-form VAR innovations, $\boldsymbol{\Sigma} = E(\boldsymbol{u}_t\boldsymbol{u}_t') = \boldsymbol{A}_0^{-1}(\boldsymbol{A}_0^{-1})'$ with $\boldsymbol{u}_t = \boldsymbol{y}_t - \boldsymbol{B}\boldsymbol{x}_t$, and $\boldsymbol{Q}$ is an $n \times n$ orthonormal matrix (i.e. $\boldsymbol{Q}\boldsymbol{Q}' = \boldsymbol{I}_n$). The reduced-form parameters are denoted by $\boldsymbol{\phi} = (\text{vec}(\boldsymbol{B})', \text{vech}(\boldsymbol{\Sigma}_{tr})')' \in \boldsymbol{\Phi}$ and the space of $n \times n$ orthonormal matrices by $\mathcal{O}(n)$.

The impulse responses to standard-deviation shocks are obtained from the coefficients of the vector moving average representation of the VAR:

$$\boldsymbol{y}_t = \sum_{h=0}^{\infty} \boldsymbol{C}_h \boldsymbol{\Sigma}_{tr}\boldsymbol{Q}\boldsymbol{\varepsilon}_{t-h},$$

where $\boldsymbol{C}_h$ is defined recursively by $\boldsymbol{C}_h = \sum_{l=1}^{\min\{h,p\}} \boldsymbol{B}_l\boldsymbol{C}_{h-l}$ for $h \geq 1$ with $\boldsymbol{C}_0 = \boldsymbol{I}_n$. The (i,j)th element of the matrix $\boldsymbol{C}_h\boldsymbol{\Sigma}_{tr}\boldsymbol{Q}$ is the horizon-h impulse response of the ith variable to the jth structural shock, denoted by $\eta_{i,j,h}(\boldsymbol{\phi}, \boldsymbol{Q}) = \boldsymbol{c}'_{ih}(\boldsymbol{\phi})\boldsymbol{q}_j$, where $\boldsymbol{c}'_{ih}(\boldsymbol{\phi}) = \boldsymbol{e}'_{i,n}\boldsymbol{C}_h\boldsymbol{\Sigma}_{tr}$ is the ith row of $\boldsymbol{C}_h\boldsymbol{\Sigma}_{tr}$ and $\boldsymbol{q}_j = \boldsymbol{Q}\boldsymbol{e}_{j,n}$ is the jth column of $\boldsymbol{Q}$. The horizon-h impulse response of the ith variable to a shock in the first variable that raises the first variable by one unit on impact is then

$$\tilde{\eta}_{i,1,h}(\boldsymbol{\phi}, \boldsymbol{Q}) = \frac{\eta_{i,1,h}(\boldsymbol{\phi}, \boldsymbol{Q})}{\eta_{1,1,0}(\boldsymbol{\phi}, \boldsymbol{Q})} = \frac{\boldsymbol{c}'_{ih}(\boldsymbol{\phi})\boldsymbol{q}_1}{\mathbf{e}'_{1,n}\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1},$$

which is well-defined whenever $\eta_{1,1,0}(\boldsymbol{\phi}, \boldsymbol{Q}) \neq 0$. In what follows, I refer to this parameter as an 'impulse response to a unit shock' and the impulse response in the denominator as the 'normalising impulse response'. The assumption that the normalising impulse response is the impact response of the first variable to the first shock is made to ease notation. In some contexts, it may be natural to normalise the impulse responses such that a specific variable increases by one unit at some longer (non-impact) horizon; for example, when estimating the effects of news shocks, the natural normalising variable may not respond at shorter horizons. The discussion below generalises straightforwardly to this more general setting.

2.2 Identifying Restrictions and Identified Sets

Imposing identifying restrictions on functions of the structural parameters is equivalent to imposing restrictions on $\boldsymbol{Q}$ given $\boldsymbol{\phi}$; for example, consider a sign restriction on an impulse response such that $\eta_{i,j,h}(\boldsymbol{\phi}, \boldsymbol{Q}) = \boldsymbol{c}'_{ih}(\boldsymbol{\phi})\boldsymbol{q}_j \geq 0$. This is a linear inequality restriction on $\boldsymbol{q}_j$, where the coefficients in the restriction are a function of $\boldsymbol{\phi}$. More generally, let $S(\boldsymbol{\phi}, \boldsymbol{Q}) \geq \mathbf{0}_{s\times 1}$ represent a collection of s sign restrictions (including the sign normalisation $\text{diag}(\boldsymbol{A}_0) \geq \mathbf{0}_{n\times 1}$). Similarly, represent a collection of f

zero restrictions by $F(\boldsymbol{\phi},\boldsymbol{Q}) = \mathbf{0}_{f\times 1}$. For example, these could include zero restrictions on impulse responses, elements of $\boldsymbol{A}_0$ or long-run cumulative impulse responses.[7]

Let f_i represent the number of zero restrictions constraining the ith column of $\boldsymbol{Q}$ with $\sum_{i=1}^{n} f_i = f$. I assume that the variables are ordered such that f_i is weakly decreasing and that $f_i \leq n-i$ for $i = 1,\ldots,n$ with strict inequality for at least one i; this is a sufficient condition for the model to be set-identified (Rubio-Ramírez *et al* 2010; Bacchiocchi and Kitagawa 2021).

Given a collection of sign and zero restrictions, the identified set for $\boldsymbol{Q}$ – which collects observationally equivalent parameter values (i.e. parameter values corresponding to the same value of the likelihood function) – is

$$Q(\boldsymbol{\phi}|S,F) = \{\boldsymbol{Q} \in \mathcal{O}(n) : S(\boldsymbol{\phi},\boldsymbol{Q}) \geq \mathbf{0}_{s\times 1}, F(\boldsymbol{\phi},\boldsymbol{Q}) = \mathbf{0}_{f\times 1}\}.$$

The identified set for a particular impulse response is then the set of values of that impulse response as $\boldsymbol{Q}$ varies over its identified set; that is, $\eta_{i,j,h}(\boldsymbol{\phi}|S,F) = \{\eta_{i,j,h}(\boldsymbol{\phi},\boldsymbol{Q}) : \boldsymbol{Q} \in Q(\boldsymbol{\phi}|S,F)\}$ or $\tilde{\eta}_{i,j,h}(\boldsymbol{\phi}|S,F) = \{\tilde{\eta}_{i,j,h}(\boldsymbol{\phi},\boldsymbol{Q}) : \boldsymbol{Q} \in Q(\boldsymbol{\phi}|S,F)\}$. Note that identified sets may be empty.

2.3 Robust Bayesian Inference in Set-identified SVARs

The standard approach to conducting Bayesian inference in set-identified SVARs involves specifying a prior for the reduced-form parameters $\boldsymbol{\phi}$ and a uniform prior for the orthonormal matrix $\boldsymbol{Q}$ (Uhlig 2005; Rubio-Ramírez *et al* 2010; Arias *et al* 2018). To draw from the resulting posterior in practice, one samples values of $\boldsymbol{\phi}$ from its posterior and $\boldsymbol{Q}$ from a uniform distribution over $Q(\boldsymbol{\phi}|F)$ and discards draws that violate the sign restrictions. Assume there is a scalar parameter of interest that is a function of the structural parameters, $\eta \equiv \eta(\boldsymbol{\phi},\boldsymbol{Q})$ (e.g. a particular impulse response). Draws of η are obtained by transforming the draws of $\boldsymbol{\phi}$ and $\boldsymbol{Q}$, and the posterior is summarised using quantities such as the posterior mean and quantiles.

Let $\pi_{\boldsymbol{\phi}}$ be a prior for $\boldsymbol{\phi} \in \boldsymbol{\Phi}$, where $\boldsymbol{\Phi}$ is the space of reduced-form parameters such that $Q(\boldsymbol{\phi}|S,F)$ is nonempty. A joint prior for the full set of parameters $\boldsymbol{\theta} = (\boldsymbol{\phi}',\text{vec}(\boldsymbol{Q})')'$ can be decomposed as $\pi_{\boldsymbol{\theta}} = \pi_{\boldsymbol{Q}|\boldsymbol{\phi}}\pi_{\boldsymbol{\phi}}$, where $\pi_{\boldsymbol{Q}|\boldsymbol{\phi}}$ is the conditional prior for $\boldsymbol{Q}$ given $\boldsymbol{\phi}$ (which assigns zero prior density outside of $Q(\boldsymbol{\phi}|S,F)$). After observing the data $\boldsymbol{Y}$, the posterior is $\pi_{\boldsymbol{\theta}|\boldsymbol{Y}} = \pi_{\boldsymbol{\phi}|\boldsymbol{Y}}\pi_{\boldsymbol{Q}|\boldsymbol{\phi}}$, where $\pi_{\boldsymbol{\phi}|\boldsymbol{Y}}$ is the posterior for $\boldsymbol{\phi}$. The prior for $\boldsymbol{\phi}$ is updated by the data (through the likelihood), whereas the conditional prior for $\boldsymbol{Q}$ given $\boldsymbol{\phi}$ is not, because $\boldsymbol{Q}$ does not appear in the likelihood. This raises the concern that posterior inferences may be sensitive to the choice of $\pi_{\boldsymbol{Q}|\boldsymbol{\phi}}$, and suggests that it may be important for researchers to assess or eliminate this sensitivity.[8]

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7 See Stock and Watson (2016) or Kilian and Lütkepohl (2017) for overviews of approaches to identification in SVARs. See Giacomini and Kitagawa (2021) for more information about the form of the mappings $S(\boldsymbol{\phi},\boldsymbol{Q})$ and $F(\boldsymbol{\phi},\boldsymbol{Q})$ under different types of identifying restrictions.

8 Inoue and Kilian (2022), Kilian (forthcoming) and Rubio-Ramírez (forthcoming) argue that posterior sensitivity to the choice of prior is typically not quantitatively important in SVAR applications. However, the evidence that they cite is based on comparing prior and posterior distributions of impulse responses. As discussed in Poirier (1998) and Giacomini Kitagawa and Read (2021b, forthcoming), this comparison is not informative about posterior sensitivity when models are set-identified; instead, the relevant measure of posterior sensitivity is the extent to which the posterior changes when the unrevisable component of the prior changes.

To this end, I adopt the 'robust' (multiple-prior) Bayesian approach to inference in set-identified models proposed by Giacomini and Kitagawa (2021). In the context of an SVAR, this approach eliminates the source of posterior sensitivity arising due to the fact that $\pi_{\boldsymbol{Q}|\boldsymbol{\phi}}$ is never updated. The key feature of the approach is that it replaces $\pi_{\boldsymbol{Q}|\boldsymbol{\phi}}$ with the class of all conditional priors that are consistent with the identifying restrictions:

$$\Pi_{\boldsymbol{Q}|\boldsymbol{\phi}} = \left\{\pi_{\boldsymbol{Q}|\boldsymbol{\phi}} : \pi_{\boldsymbol{Q}|\boldsymbol{\phi}}\big(\mathcal{Q}(\boldsymbol{\phi}|S,F)\big) = 1\right\}.$$

Combining the class of priors, $\Pi_{\boldsymbol{Q}|\boldsymbol{\phi}}$, with $\pi_{\boldsymbol{\phi}|\boldsymbol{Y}}$ generates a class of posteriors for $\boldsymbol{\theta}$:

$$\Pi_{\boldsymbol{\theta}|\boldsymbol{Y}} = \left\{\pi_{\boldsymbol{\theta}|\boldsymbol{Y}} = \pi_{\boldsymbol{Q}|\boldsymbol{\phi}}\pi_{\boldsymbol{\phi}|\boldsymbol{Y}} : \pi_{\boldsymbol{Q}|\boldsymbol{\phi}} \in \Pi_{\boldsymbol{Q}|\boldsymbol{\phi}}\right\}.$$

The class of posteriors for $\boldsymbol{\theta}$ induces a class of posteriors for η, $\Pi_{\eta|\boldsymbol{Y}}$. Giacomini and Kitagawa (2021) suggest summarising $\Pi_{\eta|\boldsymbol{Y}}$ by reporting the 'set of posterior means':

$$\left[\int_{\boldsymbol{\Phi}} \ell(\boldsymbol{\phi}) d\pi_{\boldsymbol{\phi}|\boldsymbol{Y}}, \int_{\boldsymbol{\Phi}} u(\boldsymbol{\phi}) d\pi_{\boldsymbol{\phi}|\boldsymbol{Y}}\right],$$

where $\ell(\boldsymbol{\phi}) = \inf\{\eta(\boldsymbol{\phi},\boldsymbol{Q}) : \boldsymbol{Q} \in \boldsymbol{\mathcal{Q}}(\boldsymbol{\phi}|S,F)\}$ is the lower bound of the identified set for η and $u(\boldsymbol{\phi}) = \sup\{\eta(\boldsymbol{\phi},\boldsymbol{Q}) : \boldsymbol{Q} \in \boldsymbol{\mathcal{Q}}(\boldsymbol{\phi}|S,F)\}$ is the upper bound. The set of posterior means is an interval that contains all posterior means corresponding to the posteriors in $\Pi_{\eta|\boldsymbol{Y}}$.[9] They also suggest reporting a robust credible region with credibility level α, which is an interval estimate for η such that the posterior probability put on the interval is at least α for all posteriors in $\Pi_{\eta|\boldsymbol{Y}}$. Additionally, the class of posteriors generates a set of posterior probabilities assigned to any given hypothesis (e.g. that the output response to a monetary policy shock is negative at some horizon). This set can be summarised by the posterior lower and upper probabilities, which are, respectively, the smallest and largest posterior probabilities assigned to the hypothesis over all posteriors in $\Pi_{\eta|\boldsymbol{Y}}$.

3. The Unit-effect Normalisation in a Bivariate SVAR

To illustrate the issues that arise when conducting inference about impulse responses to a unit shock, I consider the simplest possible SVAR – a bivariate SVAR with no dynamics – identified using sign restrictions on impulse responses. This allows me to analytically derive identified sets for the impulse responses. See Appendix A for derivations of the results in this section.

The simplified model is $\boldsymbol{A}_0\boldsymbol{y}_t = \boldsymbol{\varepsilon}_t$, where $\boldsymbol{y}_t = (y_{1t}, y_{2t})'$, $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ and $E(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t') = \boldsymbol{I}_2$. The orthogonal reduced form of this model is $\boldsymbol{y}_t = \boldsymbol{\Sigma}_{tr}\boldsymbol{Q}\boldsymbol{\varepsilon}_t$, where $\boldsymbol{\Sigma}_{tr}$ is the lower-triangular Cholesky factor of $\boldsymbol{\Sigma} = E(\boldsymbol{y}_t\boldsymbol{y}_t')$ and $\boldsymbol{Q}$ is a 2×2 orthonormal matrix. I denote the reduced-form parameter as $\boldsymbol{\phi} = \text{vech}(\boldsymbol{\Sigma}_{tr}) = (\sigma_{11}, \sigma_{21}, \sigma_{22})'$ with $\sigma_{11}, \sigma_{22} > 0$. In the bivariate case, the space of 2×2 orthonormal matrices can be represented as

$$\mathcal{O}(2) = \left\{\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} : \theta \in [-\pi,\pi]\right\} \cup \left\{\begin{bmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{bmatrix} : \theta \in [-\pi,\pi]\right\},$$

9 The set of posterior means is always an interval regardless of whether the identified sets are convex.

where the first set is the set of 'rotation' matrices and the second is the set of 'reflection' matrices. Henceforth, I leave it implicit that $\theta \in [-\pi,\pi]$.[10]

In the absence of any identifying restrictions, the identified set for $\boldsymbol{A}_0^{-1}$ (the matrix of impact impulse responses) is

$$\boldsymbol{A}_0^{-1} \in \left\{\begin{bmatrix} \sigma_{11}\cos\theta & -\sigma_{11}\sin\theta \\ \sigma_{21}\cos\theta+\sigma_{22}\sin\theta & \sigma_{22}\cos\theta-\sigma_{21}\sin\theta \end{bmatrix}\right\} \cup \left\{\begin{bmatrix} \sigma_{11}\cos\theta & \sigma_{11}\sin\theta \\ \sigma_{21}\cos\theta+\sigma_{22}\sin\theta & \sigma_{21}\sin\theta-\sigma_{22}\cos\theta \end{bmatrix}\right\},$$

and the identified set for $\boldsymbol{A}_0$ (the matrix of structural coefficients) is

$$\boldsymbol{A}_0 \in \left\{\frac{1}{\sigma_{11}\sigma_{22}}\begin{bmatrix} \sigma_{22}\cos\theta-\sigma_{21}\sin\theta & \sigma_{11}\sin\theta \\ -\sigma_{21}\cos\theta-\sigma_{22}\sin\theta & \sigma_{11}\cos\theta \end{bmatrix}\right\} \cup \left\{\frac{1}{\sigma_{11}\sigma_{22}}\begin{bmatrix} \sigma_{22}\cos\theta-\sigma_{21}\sin\theta & \sigma_{11}\sin\theta \\ \sigma_{21}\cos\theta+\sigma_{22}\sin\theta & -\sigma_{11}\cos\theta \end{bmatrix}\right\}.$$

Throughout, I impose the 'sign normalisation' $\mathrm{diag}(\boldsymbol{A}_0) \geq \mathbf{0}_{2\times 1}$.[11]

Consider the case where the impact response of the first variable to the first shock is restricted to be nonnegative ($\eta_{1,1,0} \equiv \mathbf{e}_{1,2}'\mathbf{A}_0^{-1}\mathbf{e}_{1,2} \geq 0$) and the impact response of the second variable to the first shock is restricted to be nonpositive ($\eta_{2,1,0} \equiv \mathbf{e}_{2,2}'\mathbf{A}_0^{-1}\mathbf{e}_{1,2} \leq 0$). The identifying restrictions generate an identified set for θ, which can in turn be used to obtain an identified set for $\eta_{1,1,0}$:

$$\eta_{1,1,0} \in \begin{cases} \left[\sigma_{11}\cos\left(\arctan\left(\min\left\{\frac{\sigma_{22}}{\sigma_{21}},\frac{\sigma_{21}}{\sigma_{22}}\right\}\right)\right),\sigma_{11}\right] & \text{if } \sigma_{21}<0 \\ \left[0,\sigma_{11}\cos\left(\arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right)\right] & \text{if } \sigma_{21}\geq 0. \end{cases}$$

The identified set for $\eta_{1,1,0}$ excludes zero when $\sigma_{21}<0$, but it includes zero when $\sigma_{21}\geq 0$. The sign restrictions therefore cannot rule out the possibility that a structural shock to the first variable results in no change in the first variable.

The impulse response of the second variable to a unit shock in the first variable is

$$\tilde{\eta}_{2,1,0} \equiv \frac{\eta_{2,1,0}}{\eta_{1,1,0}} = \frac{\sigma_{21}\cos\theta+\sigma_{22}\sin\theta}{\sigma_{11}\cos\theta} = \frac{\sigma_{21}}{\sigma_{11}}+\frac{\sigma_{22}}{\sigma_{11}}\tan\theta.$$

The identified set for this impulse response is

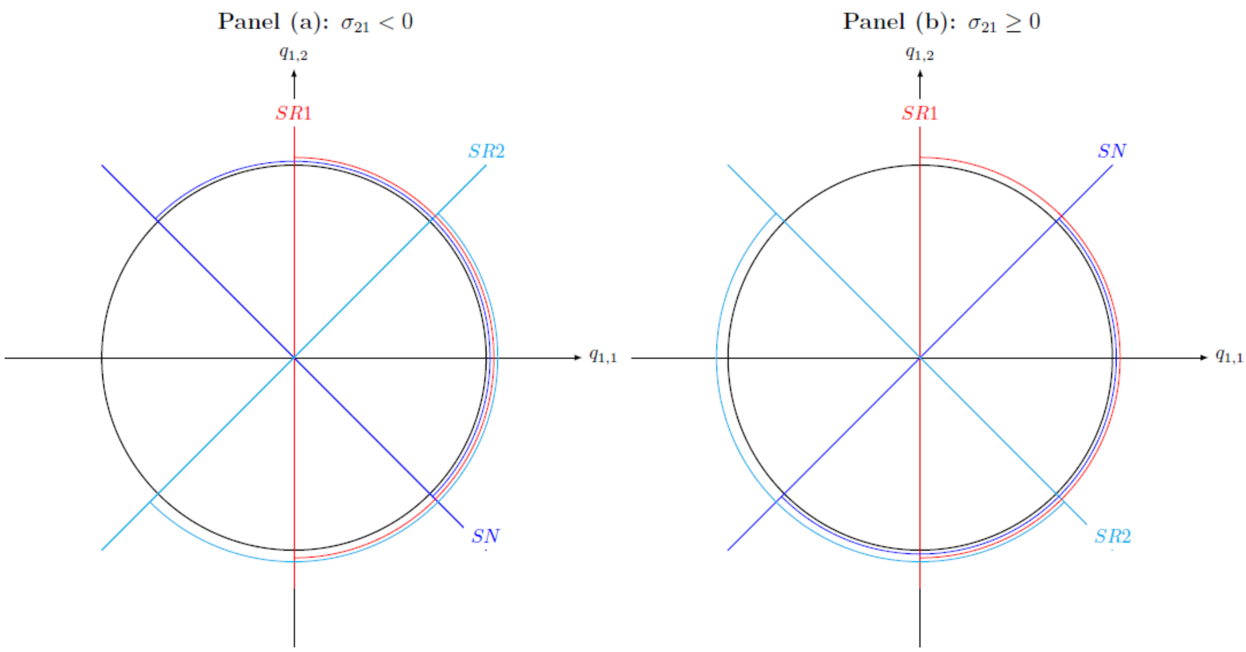
$$\tilde{\eta}_{2,1,0} \in \begin{cases} \left[\frac{\sigma_{21}}{\sigma_{11}}+\frac{\sigma_{22}^2}{\sigma_{11}\sigma_{21}},0\right] & \text{if } \sigma_{21}<0 \\ (-\infty,0] & \text{if } \sigma_{21}\geq 0. \end{cases}$$

10 Baumeister and Hamilton (2015) use a similar example to show that the standard uniform prior over $\boldsymbol{Q}$ is informative about impulse responses. In particular, the implicit prior over the impulse response to a unit shock is a Cauchy distribution that is truncated by the sign restrictions, where the points of truncation depend on the reduced-form parameters. In contrast to their example, my example focuses on the *identified set* for the impulse response to a unit shock as the object of interest.

11 If the bivariate system were interpreted as a model of supply and demand, the sign normalisation imposes that positive supply and demand shocks represent outward shifts in the supply and demand curves, respectively.

When $\sigma_{21} < 0$, the lower bound of this identified set is negative and finite, while the upper bound is zero. The identified set for $\tilde{\eta}_{2,1,0}$ is therefore bounded. In contrast, when $\sigma_{21} \geq 0$, the identified set for $\tilde{\eta}_{2,1,0}$ is unbounded (below); $\tilde{\eta}_{2,1,0}$ diverges to $-\infty$ as θ approaches $-\pi/2$ (which is the lower bound of the identified set for θ) from above, which is equivalent to $\eta_{1,1,0}$ approaching zero from above. The upper bound of the identified set for this impulse response is equal to zero, so the sign restrictions are completely uninformative about $\tilde{\eta}_{2,1,0}$ outside of its sign (which is imposed).

Figure 1: Identified Sets for q_1 in Bivariate Model



Notes: This figure depicts the identified set for $\boldsymbol{q}_1 = (q_{1,1}, q_{1,2})'$ under the sign normalisation and sign restrictions described in the text. The black circle is the unit circle. The straight lines represent the boundaries of the half-spaces generated by the identifying restrictions: 'SN' corresponds to the sign normalisation $\boldsymbol{e}_{1,2}'\boldsymbol{A}_0\boldsymbol{e}_{1,2} \geq 0$; 'SR1' corresponds to the sign restriction $\eta_{1,1,0} \geq 0$; 'SR2' corresponds to the sign restriction $\eta_{2,1,0} \leq 0$. The coloured arcs represent the sets of values of $\boldsymbol{q}_1$ satisfying each individual restriction; the arc of the unit circle where the three coloured arcs overlap is the identified set for $\boldsymbol{q}_1$.

Figure 1 provides some alternative geometric intuition behind this result. Given that the identifying restrictions constrain $\boldsymbol{q}_1$ only, they can be represented as three half-spaces (corresponding to the sign normalisation plus the two sign restrictions on impulse responses) in two-dimensional space (Figure 1).[12] The identified set for $\boldsymbol{q}_1$ is given by the intersection of these half-spaces with the unit circle. When this identified set includes the boundary of the half-space corresponding to the sign restriction on the impact response of the first variable to the first shock, $\eta_{1,1,0} \geq 0$, the impulse response of the second variable to a unit shock in the first variable, $\tilde{\eta}_{2,1,0}$, can be made arbitrarily large by considering a sequence for $\boldsymbol{q}_1$ (equivalently, θ) converging to the point of singularity, $\eta_{1,1,0} = 0$. Whether it is possible to do this depends on the sign of σ_{21}; when $\sigma_{21} < 0$, the intersection of the

12 Similar graphical illustrations are presented in Granziera *et al* (2018) and Amir-Ahmadi and Drautzburg (2021).

half-spaces always excludes this point of singularity (Panel (a)); when $\sigma_{21} \geq 0$, the point of singularity is included (Panel (b)).[13]

Note that these results are not dependent on the fact that the sign restrictions are imposed on only a single column of $\boldsymbol{Q}$. Under the additional restrictions that the impulse responses of both variables to the second shock are positive, the identified set for $\eta_{1,1,0}$ continues to include zero when $\sigma_{21} \geq 0$ and the identified set for $\tilde{\eta}_{2,1,0}$ is unbounded (see Appendix A.2 for details). The results are also not dependent on imposing the sign restrictions with weak, rather than strict, inequality. Replacing the set of sign restrictions (including the sign normalisations) with strict inequalities yields identified sets that are open, rather than closed, intervals, but the identified set for $\tilde{\eta}_{2,1,0}$ remains unbounded when $\sigma_{21} \geq 0$.

This example highlights that the identified set for the impulse response to a unit shock may be unbounded if the identified set for the impact response of the normalising variable to a standard-deviation shock includes zero. The exercise also demonstrates that sign restrictions may or may not be informative about impulse responses to a unit shock. Whether this is the case depends on the values of the reduced-form parameters. The following sections discuss some implications of unboundedness for conducting inference about the impulse responses to a unit shock.[14]

3.1 Robust Bayesian Inference Under Unboundedness

For ease of exposition, I make the simplifying assumption that $\pi_{\boldsymbol{\phi}|\boldsymbol{Y}}$ is supported only on two values of the reduced-form parameters: $\boldsymbol{\phi}^a = (\sigma_{11}, \sigma_{21}^a, \sigma_{22})'$ and $\boldsymbol{\phi}^b = (\sigma_{11}, \sigma_{21}^b, \sigma_{22})'$, where $\sigma_{21}^a < 0 \leq \sigma_{21}^b$. I denote the lower bound of the identified set for $\tilde{\eta}_{2,1,0}$ when $\boldsymbol{\phi} = \boldsymbol{\phi}^a$ by $L(\boldsymbol{\phi}^a)$ and the posterior probability that $\sigma_{21} = \sigma_{21}^a$ by α. Under this assumption, the identified set for $\tilde{\eta}_{2,1,0}$ is $[L(\boldsymbol{\phi}^a), 0]$ with posterior probability α and it is $(-\infty, 0]$ with posterior probability $1 - \alpha$.

The set of posterior means, which has bounds equal to the posterior means of the bounds of the identified set, will be $(-\infty, 0]$ unless $\alpha = 1$. Consequently, if $\pi_{\boldsymbol{\phi}|\boldsymbol{Y}}$ places positive posterior probability on the event $\sigma_{21} = \sigma_{21}^b \geq 0$, the set of posterior means is completely uninformative about the impulse response to a unit shock (other than its sign, which is imposed by the sign restrictions).

The median of the upper bound of the identified set is zero regardless of the value of α. When $\alpha \geq 0.5$, the posterior median of the lower bound of the identified set is $L(\boldsymbol{\phi}^a)$. The set of posterior medians – which is an interval with lower (upper) bound equal to the posterior median of the lower (upper) bound of the identified set – will therefore be bounded despite the set of posterior means being unbounded. In contrast, when $\alpha < 0.5$, the posterior median of the lower bound of the identified set is $-\infty$, so the set of posterior medians is unbounded. By similar logic, the set of posterior τ-quantiles will be bounded so long as $\alpha \geq \tau$. The class of posteriors may therefore still

13 When imposing only the sign normalisation, the identified set for $\tilde{\eta}_{2,1,0}$ is always $(-\infty, \infty)$; in this case, it is possible to approach the point of singularity $\eta_{1,1,0} = 0$ from the positive or negative direction regardless of the value of σ_{21}. When imposing the sign normalisation and the sign restriction $\eta_{1,1,0} \geq 0$, the identified set for $\tilde{\eta}_{2,1,0}$ is always $(-\infty, 0]$; in this case, the point of singularity can be approached from the positive direction only.

14 Identified sets can also be unbounded when the parameter of interest is the structural coefficient on a particular variable after normalising the coefficient on another variable to equal unity (i.e. the ratio of elements of $\boldsymbol{A}_0$). If the identified set for the normalising coefficient includes zero, the identified set for the ratio of coefficients may be unbounded.

contain useful information about particular posterior quantiles even when the identified set is unbounded with positive posterior probability.

A robust credible interval with credibility $1-\tau$ can be constructed by taking the $\tau/2$ quantile of $\ell(\boldsymbol{\phi})$ and the $1-\tau/2$ quantile of $u(\boldsymbol{\phi})$. Whether the robust credible interval is bounded will therefore depend on the credibility level and α. In particular, boundedness of the robust credible interval requires that the sets of $\tau/2$ and $1-\tau/2$ quantiles are both bounded, which will be the case in the current example if $\alpha \geq \tau/2$.

Consider the hypothesis that $\tilde{\eta}_{2,1,0} \leq x$ for some $x < 0$. The posterior *lower* probability of this hypothesis is equal to the posterior probability that the identified set is contained within the interval $(-\infty, x]$. This probability is zero for all $x < 0$. The posterior *upper* probability of the hypothesis is equal to the posterior probability that the identified set intersects the interval $(-\infty, x]$. The posterior upper probability is one for $L(\boldsymbol{\phi}^a) \leq x < 0$ and is $1-\alpha$ for $x < L(\boldsymbol{\phi}^a)$. The set of posterior probabilities for the hypothesis $\tilde{\eta}_{2,1,0} \leq x$ is therefore $[0,1]$ for $L(\boldsymbol{\phi}^a) \leq x < 0$ and is $[0,1-\alpha]$ for $x < L(\boldsymbol{\phi}^a)$. As α approaches zero, so that the identified set is almost always unbounded, the set of posterior probabilities converges to the unit interval for all values of x. In this case, the sign restrictions are not informative about the hypothesis regardless of the value of x. In contrast, as α approaches one, the set of posterior probabilities converges to zero for sufficiently negative values of x (i.e. for $x < L(\boldsymbol{\phi}^a)$). In this case, we can conclude that 'large' responses are assigned low posterior probability regardless of the choice of conditional prior.

This discussion illustrates that it is still possible to extract information about the impulse responses to a unit shock using the robust Bayesian approach to inference when the identified set is unbounded with positive posterior probability. The takeaways from this stylised model extend to the general case of an n-dimensional SVAR with dynamics and/or where the posterior for $\boldsymbol{\phi}$ has continuous support.

3.1.1 Frequentist validity of robust Bayesian approach

For general set-identified models, Giacomini and Kitagawa (2021) provide high-level conditions under which their robust Bayesian approach to inference has a valid frequentist interpretation, in the sense that the set of posterior means is consistent for the true identified set (i.e. the identified set when $\boldsymbol{\phi}$ is equal to its true value, $\boldsymbol{\phi}_0$) and the robust credible interval has correct frequentist coverage for the true identified set. In the context of SVARs and when the parameter of interest is an impulse response to a standard-deviation shock, Giacomini and Kitagawa (2021) provide sufficient conditions under which these high-level conditions will hold. In particular, the set of posterior means can be interpreted as a consistent estimator of the true identified set if the identified set is convex and continuous at $\boldsymbol{\phi} = \boldsymbol{\phi}_0$. Additionally, if the endpoints of the identified set ($\ell(\boldsymbol{\phi})$ and $u(\boldsymbol{\phi})$) are differentiable in $\boldsymbol{\phi}$ at $\boldsymbol{\phi} = \boldsymbol{\phi}_0$ with nonzero derivatives, the robust credible interval has valid frequentist coverage of the true identified set.

When the parameter of interest is an impulse response to a unit shock, the high-level conditions for frequentist validity of the robust Bayesian approach are not necessarily satisfied. For example, these conditions include the assumption that the true identified set is bounded. Consequently, the robust Bayesian approach to inference is not guaranteed to have an asymptotically valid frequentist interpretation when the parameter of interest is an impulse response to a unit shock.

To illustrate, consider the bivariate model and assume that $\boldsymbol{\phi}_0$ is such that $\sigma_{21} \geq 0$, so the true identified set is unbounded. For values of $\boldsymbol{\phi}$ in a small neighbourhood of $\boldsymbol{\phi}_0$, $l(\boldsymbol{\phi}) = -\infty$ and $u(\boldsymbol{\phi}) = 0$, so naively applying the robust Bayesian approach in this case will (asymptotically) yield a robust credible interval of $(-\infty, 0]$. Clearly, this interval always (weakly) includes the true identified set, so the asymptotic frequentist coverage probability will be trivially equal to one, which is greater than the nominal credibility level τ (i.e. the robust credible interval is conservative).[15]

3.2 Frequentist Estimation Under Unboundedness

Unboundedness may also arise when estimating or conducting inference about impulse responses to a unit shock in a frequentist framework. Let $\widehat{\boldsymbol{\phi}} = (\hat{\sigma}_{11}, \hat{\sigma}_{21}, \hat{\sigma}_{22})'$ be the MLE of $\boldsymbol{\phi}$. In the current bivariate example, if $\widehat{\boldsymbol{\phi}}$ is such that $\hat{\sigma}_{21} < 0$, the frequentist estimate of the identified set for $\tilde{\eta}_{2,1,0}$ – which simply plugs the MLE of $\widehat{\boldsymbol{\phi}}$ into the expression for the identified set given in Section 3.1 – will be bounded. In contrast, if $\widehat{\boldsymbol{\phi}}$ is such that $\hat{\sigma}_{21} \geq 0$, a frequentist estimate of the identified set for $\tilde{\eta}_{2,1,0}$ will be unbounded.

4. Checking for Unboundedness in SVARs

As noted above, the lessons from the bivariate model of Section 3 extend to the general setting of an n-dimensional SVAR with dynamics. They also extend to the case where there are both sign and zero restrictions on the structural parameters. In this general setting, analytical expressions for identified sets are not usually available and it is necessary to approximate the bounds of the identified set numerically. This section explains how to check whether the identified sets for the impulse responses to a unit shock may be unbounded in this setting.

Checking whether the identified set is unbounded is helpful for understanding whether particular inferential outputs (e.g. sets of posterior means or quantiles) are themselves unbounded. From a practical standpoint, it is also important to check whether the identified set is unbounded to understand the properties of numerical approximations of the identified set. One approach to computing the bounds of the identified set is to use a numerical optimisation routine where the objective function to be minimised or maximised is $\tilde{\eta}_{i,j,h}(\boldsymbol{\phi}, \boldsymbol{Q})$ and the constraints are the set of identifying restrictions. If the identified set is unbounded, standard gradient-based numerical optimisation routines (e.g. an interior-point algorithm) will terminate at some large, but arbitrary, value of the objective function. Another approach to computing the bounds is to obtain many random draws of $\boldsymbol{Q}$ from a distribution over $\mathcal{Q}(\boldsymbol{\phi}|S, F)$ (e.g. a uniform distribution) and compute the minimum and maximum over these draws. When the identified set is bounded, the approximation error from this approach will vanish as the number of draws increases, but this will not be the case when the identified set is unbounded.

In the n-variable SVAR (described in Section 2), assume that the sign restrictions $S(\boldsymbol{\phi}, \boldsymbol{Q}) \geq \mathbf{0}_{s\times 1}$ include the restriction that the impact response of the first variable to the first shock is nonnegative, $\eta_{1,1,0} = \boldsymbol{e}'_{1,n}\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 \geq 0$. For example, in the context of estimating the effects of monetary policy

15 In the current bivariate example and when $\boldsymbol{\phi}_0$ is such that $\sigma_{21} < 0$, the true identified set is bounded, but the robust credible interval has an asymptotic frequentist coverage probability equal to $1 - \tau/2 > 1 - \tau$ (i.e. the robust credible interval is conservative). This arises because the upper bound of the identified set is degenerate and is therefore not differentiable in $\boldsymbol{\phi}$ with non-zero derivative. When there are additional sign restrictions on the impulse responses to the second shock, both the lower and upper bound are differentiable in $\boldsymbol{\phi}$ at $\boldsymbol{\phi} = \boldsymbol{\phi}_0$ (see Appendix A.2), and the robust credible interval has correct coverage asymptotically.

shocks, this restriction would require that a positive monetary policy shock (the first shock) does not decrease the federal funds rate (the first variable) on impact. Such a restriction seems natural. The identified set for $\tilde{\eta}_{i,1,h}$, $(i,h) \neq (1,0)$, will be unbounded only if the identified set for $\eta_{1,1,0}$ includes zero.[16] This will be the case if there exists $\boldsymbol{Q}$ satisfying the zero restrictions, the 'binding' sign restriction on $\eta_{1,1,0}$ ($\boldsymbol{e}'_{1,n}\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 = 0$), and any remaining sign restrictions. The following proposition formalises this claim.

Proposition 4.1. *(Necessary condition for unbounded identified sets.) Assume* $Q(\boldsymbol{\phi}|S,F)$ *is nonempty and interest is in the impulse response to a unit shock in the first variable at some fixed and finite horizon* h*. The identified set for the impulse response to a unit shock to the first variable,* $\tilde{\eta}_{i,1,h}(\boldsymbol{\phi}|S,F)$*, is unbounded for* $(i,h) \neq (1,0)$ *only if* $0 \in \eta_{1,1,0}(\boldsymbol{\phi}|S,F)$.

Proposition 4.1 provides a necessary condition for the unboundedness of $\tilde{\eta}_{i,1,h}(\boldsymbol{\phi}|S,F)$. Intuitively, if the identified set for $\eta_{1,1,0}$ does not contain zero, it is not possible to construct a sequence for $\boldsymbol{Q}$ converging to the point where $\eta_{1,1,0} = 0$ such that $\tilde{\eta}_{i,1,h}$ diverges. If the identified set for $\eta_{1,1,0}$ includes zero, it *may* be possible to construct a sequence for $\boldsymbol{Q}$ converging to the point $\eta_{1,1,0} = 0$ such that $\tilde{\eta}_{i,1,h}$ diverges. However, the condition does not guarantee that $\tilde{\eta}_{i,1,h}(\boldsymbol{\phi}|S,F)$ is unbounded. To give an example, consider an extension of the bivariate model of Section 3 with dynamics:

$$\boldsymbol{y}_t = \boldsymbol{B}_1\boldsymbol{y}_{t-1} + \boldsymbol{\Sigma}_{tr}\boldsymbol{Q}\boldsymbol{\varepsilon}_t.$$

Assume that $\boldsymbol{B}_1$ is diagonal with diagonal elements $\mathrm{diag}(\boldsymbol{B}_1) = (b_{11}, b_{22})'$. When $\sigma_{21} \geq 0$, the identified set for $\eta_{1,1,0}$ includes zero. However, the identified set for $\tilde{\eta}_{1,1,h}$ is b_{11}^h, which is finite for any value of b_{11} and finite h.[17]

In what follows, I discuss how to check whether $\eta_{1,1,0}(\boldsymbol{\phi}|S,F)$ includes zero, in which case $\tilde{\eta}_{i,1,h}(\boldsymbol{\phi}|S,F)$ may be unbounded.

Consider imposing a set of zero and sign restrictions constraining $\boldsymbol{q}_1$ only, $F(\boldsymbol{\phi},\boldsymbol{Q}) = F(\boldsymbol{\phi})\boldsymbol{q}_1 = \boldsymbol{0}_{f\times 1}$ and $S(\boldsymbol{\phi},\boldsymbol{Q}) = S(\boldsymbol{\phi})\boldsymbol{q}_1 \geq \boldsymbol{0}_{s\times 1}$. In this setting, the following proposition states a sufficient condition for the identified set for $\eta_{1,1,0}$ to include zero.

Proposition 4.2. *(Sufficient condition for identified set for* $\eta_{1,1,0}$ *to include zero.) Assume that any sign and zero restrictions constrain* $\boldsymbol{q}_1$ *only,* $\eta_{1,1,0} = \boldsymbol{e}'_{1,n}\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 \geq 0$ *is contained within the set of sign restrictions* $S(\boldsymbol{\phi})\boldsymbol{q}_1 \geq \boldsymbol{0}_{s\times 1}$ *and the number of zero restrictions in* $F(\boldsymbol{\phi})\boldsymbol{q}_1 = \boldsymbol{0}_{f\times 1}$ *satisfies* $0 \leq f < n-1$ *with* $\mathrm{rank}(F(\boldsymbol{\phi})) = r$*. If* $s + f \leq n$*, then* $0 \in \eta_{1,1,0}(\boldsymbol{\phi}|S,F)$.

The sufficient condition in Proposition 4.2 can be used to verify whether the identified set for $\eta_{1,1,0}$ includes zero at any value of $\boldsymbol{\phi}$. The condition is easily verifiable; it simply requires counting the number of sign and zero restrictions imposed. Although the proposition only applies when the identifying restrictions constrain a single column of $\boldsymbol{Q}$, this is the case in many empirical applications; examples include Uhlig (2005) and Arias *et al* (2019) (see also the references in Gafarov *et al*

16 I abstract from the possibility of imposing sign restrictions with strict inequality (i.e. $S(\boldsymbol{\phi}) > \boldsymbol{0}_{s\times 1}$). In that case, identified sets will be an open intervals. Consequently, the identified set for $\tilde{\eta}_{i,1,h}$ could be unbounded without the identified set for $\eta_{1,1,0}$ including zero. When allowing for strict inequalities, the identified set for $\tilde{\eta}_{i,1,h}$ will be unbounded only if the closure of the identified set for $\eta_{1,1,0}$ includes zero.

17 I am indebted to Thorsten Drautzburg for suggesting this example.

(2018)). The assumption that $0 \leq f < n-1$ rules out the case where $\boldsymbol{q}_1$ (and thus any impulse response to the first shock) is point-identified.[18] If the set of sign restrictions does not include the restriction $\eta_{1,1,0} = \boldsymbol{e}_{1,n}'\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 \geq 0$, the sufficient condition for unboundedness becomes $s + f \leq n-1$; the intuition in this case is that, if $s + f \leq n-1$, we can augment the system of identifying restrictions with $\boldsymbol{e}_{1,n}'\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 \geq 0$ and then apply Proposition 4.2.

While the condition $s + f \leq n$ is unlikely to hold in applications that impose dynamic sign restrictions (i.e. sign restrictions at multiple horizons), these restrictions are not always imposed. For example, the condition is satisfied in Arias *et al* (2019), who identify a monetary policy shock by imposing sign and zero restrictions on elements of $\boldsymbol{A}_0$ (see Section 5). To identify an unconventional monetary policy shock, Gafarov *et al* (2018) impose four restrictions (one zero restriction and three signs restrictions) in a four-variable system. Beaudry, Nam and Wang (2011) identify an 'optimism' shock by imposing two restrictions (one zero restriction and one sign restriction) in a five-variable system.

When $s + f > n$, whether it is possible to construct a vector satisfying $\boldsymbol{e}_{1,n}'\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 = 0$ and the remaining identifying restrictions depends on the reduced-form parameters. Geometrically, the condition $\boldsymbol{e}_{1,n}'\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 = 0$ and the zero restrictions are jointly satisfied when $\boldsymbol{q}_1$ lies in an $(n-f-1)$-dimensional hyperplane that is orthogonal to $\boldsymbol{e}_{1,n}'\boldsymbol{\Sigma}_{tr}$ and the rows of $F(\boldsymbol{\phi})$, while the remaining sign restrictions in $S(\boldsymbol{\phi})$ require $\boldsymbol{q}_1$ to lie within the intersection of $s-1$ half-spaces. The identified set for $\eta_{1,1,0}$ will include zero if and only if the intersection of this hyperplane and these half-spaces is nonempty. When $s + f > n$, the hyperplane and half-spaces are not guaranteed to intersect; whether they intersect depends on the values of the reduced-form parameters, which determine the orientations of the hyperplane and half-spaces. To give an example using the bivariate model of Section 3, $s = 3 > 2 = n$, so the condition in Proposition 4.2 is not satisfied and zero is not necessarily included within the identified set for the normalising impulse response; in particular, zero is excluded when $\sigma_{21} < 0$ (see Figure 1 in Section 3 for a graphical illustration).

When the conditions in Proposition 4.2 do not hold, one can use numerical methods to check whether $\eta_{1,1,0}(\boldsymbol{\phi}|S,F)$ includes zero and thus whether $\tilde{\eta}_{i,1,h}(\boldsymbol{\phi}|S,F)$ may be unbounded. Let $\tilde{F}(\boldsymbol{\phi},\boldsymbol{Q}) = \boldsymbol{0}_{(f+1)\times 1}$ represent an augmented set of zero restrictions that includes a 'binding' version of the sign restriction on $\eta_{1,1,0}$ (i.e. $\boldsymbol{e}_{1,n}'\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 = 0$) and let $\tilde{S}(\boldsymbol{\phi},\boldsymbol{Q}) \geq \boldsymbol{0}_{(s-1)\times 1}$ collect the remaining sign restrictions. The identified set for $\eta_{1,1,0}$ includes zero if and only if the identified set for $\boldsymbol{Q}$ given the augmented set of restrictions, $Q(\boldsymbol{\phi}|\tilde{S},\tilde{F})$, is nonempty. In the case where the identifying restrictions constrain $\boldsymbol{q}_1$ only, the algorithms proposed in Read (forthcoming) can be used to determine whether $Q(\boldsymbol{\phi}|\tilde{S},\tilde{F})$ is nonempty.[19] When the identifying restrictions constrain multiple columns of $\boldsymbol{Q}$, one can check whether $Q(\boldsymbol{\phi}|\tilde{S},\tilde{F})$ is nonempty by drawing from a uniform distribution over $Q(\boldsymbol{\phi}|\tilde{F})$ (e.g. using the algorithms in Arias *et al* (2018) or Giacomini and Kitagawa (2021)) until a draw is obtained satisfying the remaining sign restrictions. If no such draw can be obtained after a large number of draws, this suggests that $Q(\boldsymbol{\phi}|\tilde{S},\tilde{F})$ is empty, in which case $\tilde{\eta}_{i,1,h}(\boldsymbol{\phi}|S,F)$ is bounded.

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18 When $f = n-1$ and $\mathrm{rank}(F(\boldsymbol{\phi})) = n-1$, the identified set for $\eta_{1,1,0}$ (which is a singleton) excludes zero so long as $\mathrm{rank}((F(\boldsymbol{\phi})', \boldsymbol{\Sigma}_{tr}'\boldsymbol{e}_{1,n,})') = n$. This condition would be violated in the (unrealistic) instance where the zero restrictions in $F(\boldsymbol{\phi})$ include the restriction that $\eta_{1,1,0} = 0$. Note that the condition $0 \leq f < n-1$ implicitly rules out the possibility that $n = 1$, in which case the impulse responses would be trivially point-identified.

19 If $f = n-2$, the unit-length vector $\tilde{\boldsymbol{q}}_1$ satisfying $\tilde{F}(\boldsymbol{\phi})\tilde{\boldsymbol{q}}_1 = \boldsymbol{0}_{(f+1)\times 1}$ is pinned down up to sign; such a vector can be found by computing an orthonormal basis for the null space of $\tilde{F}(\boldsymbol{\phi})$. If either $\tilde{S}(\boldsymbol{\phi})\tilde{\boldsymbol{q}}_1 \geq \boldsymbol{0}_{(s-1)\times 1}$ or $-\tilde{S}(\boldsymbol{\phi})\tilde{\boldsymbol{q}}_1 \geq \boldsymbol{0}_{(s-1)\times 1}$, then $Q(\boldsymbol{\phi}|\tilde{S},\tilde{F})$ is nonempty. For $0 \leq f < n-2$, the algorithm described in Read (forthcoming) is applicable.

5. Estimating the Effects of a 100 Basis Point Federal Funds Rate Shock

This section illustrates the empirical relevance of the issues discussed above by estimating the macroeconomic effects of a 100 basis point shock to the federal funds rate under different sets of identifying restrictions that have been used previously in the literature.

I use the reduced-form VAR considered in Uhlig (2005), Antolín-Díaz and Rubio-Ramírez (2018) and Arias *et al* (2019). The model's endogenous variables are real GDP (GDP_t), the GDP deflator ($GDPDEF_t$), a commodity price index (COM_t), total reserves (TR_t), nonborrowed reserves (NBR_t) (all in natural logarithms) and the federal funds rate (FFR_t). The data are monthly and run from January 1965 to November 2007.[20] The VAR includes 12 lags of the variables and a constant. I assume a Jeffreys' (improper) prior over the reduced-form parameters, so $\pi_{\boldsymbol{\phi}} \propto |\boldsymbol{\Sigma}|^{-(n-1)/2}$. This means that the posterior for $\boldsymbol{\phi}$ is a normal-inverse-Wishart distribution, from which it is straightforward to obtain independent draws (e.g. Del Negro and Schorfheide 2011). Under each set of restrictions below, I obtain 10,000 draws from the posterior for $\boldsymbol{\phi}$ such that the identified set is nonempty.[21] The papers listed above conduct Bayesian inference under a uniform prior for $\boldsymbol{Q}$ and primarily present impulse responses to a standard-deviation monetary policy shock.[22] In contrast, I focus on the impulse responses to a 100 basis point shock as the parameters of interest.

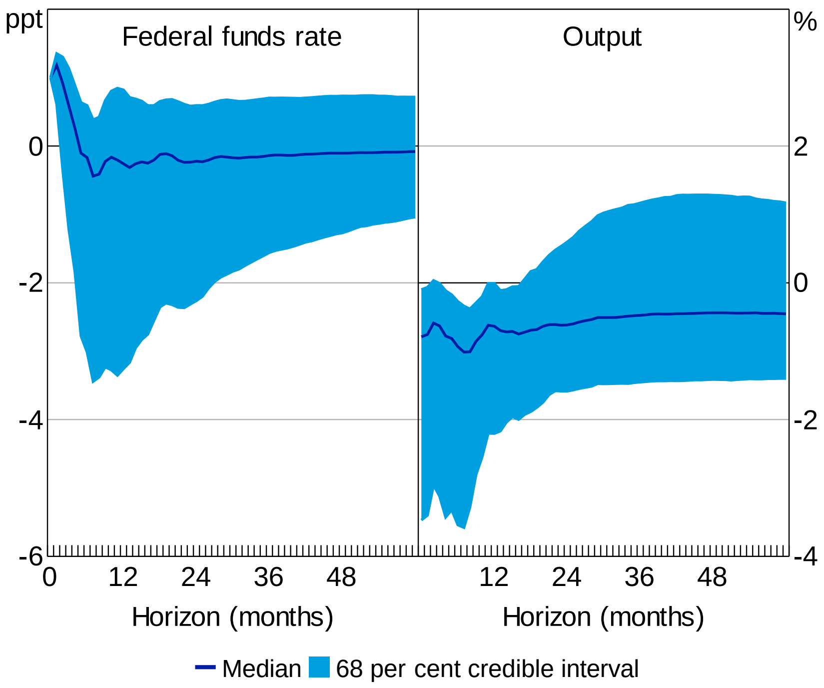
First, I consider the identifying restrictions proposed in Arias *et al* (2019), who impose sign and zero restrictions on the structural equation for the federal funds rate, which they interpret as the monetary policy reaction function. The restrictions impose that the coefficients on TR_t and NBR_t in the structural equation for FFR_t are zero, which means that the Federal Reserve does not react to changes in reserves when setting the federal funds rate. They also impose sign restrictions on the coefficients of GDP_t and $GDPDEF_t$ such that the Federal Reserve does not increase the federal funds rate in response to lower output or prices, which is consistent with the types of policy rules typically specified in New Keynesian dynamic stochastic general equilibrium (DSGE) models. Finally, the impact response of FFR_t to the monetary policy shock is restricted to be nonnegative, so that a monetary policy shock does not decrease FFR_t on impact, which seems natural. I denote this set of identifying restrictions as Restriction (1).

20 I use the updated version of the dataset from Antolín-Díaz and Rubio-Ramírez (2018). The monthly series for GDP_t and $GDPDEF_t$ are obtained by interpolation; see Arias *et al* (2019) for details.

21 When the restrictions constrain a single column of $\boldsymbol{Q}$ only (i.e. under Restrictions (1) and (2)), I check whether the identified set is nonempty at each draw of $\boldsymbol{\phi}$ using Algorithm 4.1 in Read (forthcoming).

22 Uhlig (2005) and Arias *et al* (2019) present impulse responses to a standard-deviation shock. Antolín-Díaz and Rubio-Ramírez (2018) present impulse responses that are normalised such that the median impact response of the federal funds rate is 25 basis points.

Figure 2: Impulse Responses to 100 Basis Point Shock – Restriction (1)



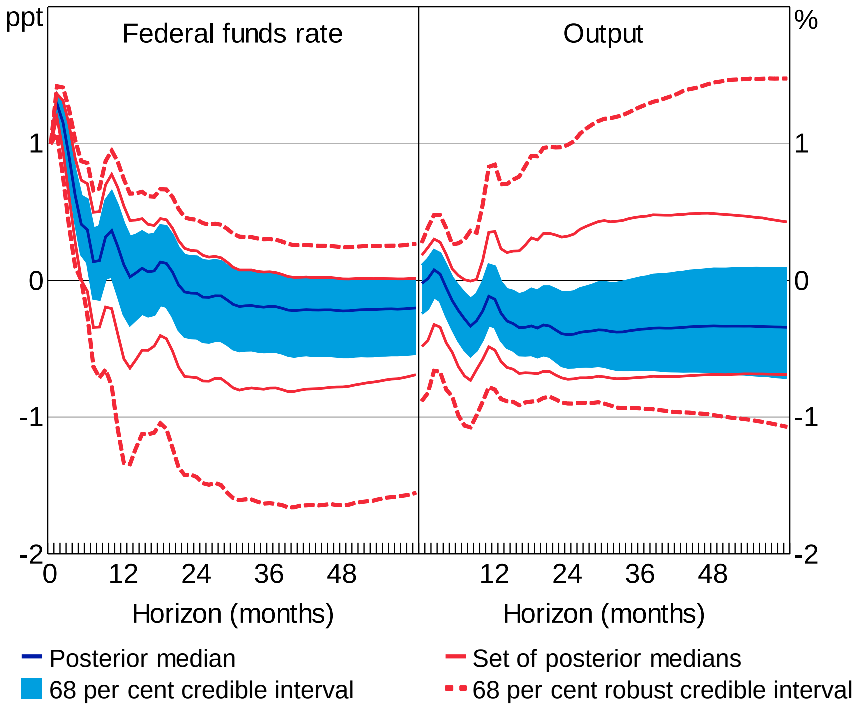
Notes: Results obtained under the identifying restrictions in Arias *et al* (2019).

Figure 2 presents the impulse responses of FFR_t and GDP_t to a 100 basis point monetary policy shock obtained under Restriction (1) and the conditionally uniform prior for $\boldsymbol{Q}$ given $\boldsymbol{\phi}$. Based on the posterior median, output falls by a maximum of about one per cent. The 68 per cent credible intervals include declines in output of close to 4 per cent, so there is considerable posterior probability assigned to very large declines in output. This set of restrictions involves four sign restrictions (including the sign normalisation on the $(1,1)$ element of $\boldsymbol{A}_0$) and two zero restrictions, so the total number of restrictions is equal to the dimension of the SVAR. This means that the sufficient condition in Proposition 4.2 of Section 4 is satisfied, so zero is always included within the identified set for the impact response of the federal funds rate; in other words, the identifying restrictions cannot rule out the possibility that the federal funds rate does not respond to a monetary policy shock on impact. Examining the approximated bounds of the identified sets for the impulse responses to a 100 basis point shock suggests that these identified sets are unbounded at every draw of $\boldsymbol{\phi}$. In turn, this means that the set of posterior means and the sets of posterior τ-quantiles are unbounded for all $\tau \in [0,1]$. Consequently, the restrictions are extremely uninformative about the impulse responses to a 100 basis point shock to the federal funds rate. An implication is that the results obtained under the conditionally uniform prior in Figure 2 are predominately driven by the choice of conditional prior rather than information in the data and identifying restrictions.

Next, I combine the restrictions from Arias *et al* (2019) with the sign restrictions on impulse responses proposed in Uhlig (2005). These sign restrictions impose that the impulse response of FFR_{t+h} to the monetary policy shock is nonnegative and the impulse responses of $GDPDEF_{t+h}$,

COM_{t+h} and NBR_{t+h} are nonpositive for $h = 0,1,\ldots,5$.[23] I refer to this set of restrictions as Restriction (2). Under a conditionally uniform prior, the additional sign restrictions appreciably tighten the posterior distribution of the impulse responses to a 100 basis point shock (Figure 3). The posterior median suggests that output falls by a maximum of 0.4 per cent about two years after the shock and the 68 per cent credible intervals no longer contain extremely large values; for example, at the two-year horizon the credible intervals span declines in output of 0.1–0.6 per cent. To what extent is the unrevisable component of the prior driving these results?

Figure 3: Impulse Responses to 100 Basis Point Shock – Restriction (2)



Notes: Results obtained under a combination of the identifying restrictions in Uhlig (2005) and Arias *et al* (2019).

Under these restrictions, there are two zero restrictions and 27 sign restrictions, so the sufficient condition in Proposition 4.2 is not satisfied. This means that the identified set for the impact response of the federal funds rate does not necessarily include zero. I therefore numerically check whether the identified set for $\eta_{1,1,0}$ includes zero at each draw from the reduced-form posterior.[24] The identified set for the impact response of the federal funds rate includes zero in only 0.06 per cent of draws from the posterior, which implies that the identified sets for the impulse responses to a unit shock are bounded close to 100 per cent of the time. Examining the numerical approximations of the identified set suggests that the identified sets for the impulse responses to a unit shock are

23 Uhlig (2005) also considers restricting the impulse responses at shorter and longer horizons than six months in robustness exercises. The choice of horizon here does not affect whether the sufficient condition in Proposition 4.2 applies. However, the proportion of draws at which the identified set for $\eta_{1,1,0}$ includes zero and the width of the sets of posterior medians and robust credible intervals vary across the different sets of restrictions. Appendix C presents additional results obtained when the impulse responses are restricted up to horizon $H \in \{2,12,24\}$.

24 Following the discussion in Section 4, I check whether $Q(\boldsymbol{\phi}|\tilde{F},\tilde{S})$ is empty using Algorithm 4.1 from Read (forthcoming), which is applicable here because only a single column of $\boldsymbol{Q}$ is restricted.

indeed unbounded at some draws of $\boldsymbol{\phi}$.[25] Since the identified sets appear to be unbounded with positive probability, the sets of posterior means for the output responses to a 100 basis point shock will also unbounded. Nevertheless, the sets of posterior medians remains bounded, because the identified sets are unbounded less in less than 50 per cent of draws. The robust credible intervals are also bounded so long as the credibility level is not too extreme (i.e. not greater than 99.4 per cent). To summarise the set of posteriors, I present the set of posterior medians and a 68 per cent robust credible interval.[26]

The set of posterior medians for the output response to a 100 basis point shock includes zero at essentially all horizons. The 68 per cent robust credible intervals for the output response include both large negative and large positive responses. Hence, under Restriction (2), there is substantial uncertainty about the output response to a 100 basis point monetary policy shock after eliminating the effect of the unrevisable component of the prior.

Finally, I add the 'narrative restrictions' proposed in Antolín-Díaz and Rubio-Ramírez (2018) to Restriction (2). I refer to this set of restrictions as Restriction (3). Narrative restrictions are restrictions on functions of the structural *shocks* in specific periods (as opposed to restrictions on functions of the structural *parameters*) that represent information about the nature of the shocks hitting the economy during particular historical episodes.[27] The specific narrative restrictions imposed are that the monetary policy shock was positive and was the 'overwhelming' contributor to the observed unexpected change (i.e. the forecast error) in the federal funds rate in October 1979. This is the month in which the Federal Reserve unexpectedly and dramatically raised the federal funds rate following Paul Volcker becoming chairman, and is widely considered an example of a monetary policy shock (e.g. Romer and Romer 1989).

The contribution of the jth structural shock to the one-step-ahead forecast error in variable i in period t is $H_{i,j,t} = \eta_{i,j,0}(\boldsymbol{\phi}, \boldsymbol{Q})\varepsilon_{j,t} = \boldsymbol{e}'_{i,n}\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_j\boldsymbol{q}'_j\boldsymbol{\Sigma}_{tr}^{-1}\boldsymbol{u}_t$. The restriction that the monetary policy shock was the 'overwhelming' contributor to the observed unexpected change in the federal funds rate means that the absolute contribution of the monetary policy shock to the forecast error in the federal funds rate is greater than the sum of the absolute contributions of all other shocks, or $|H_{i,j,t}| \geq \sum_{k\neq j}|H_{i,k,t}|$. This is a restriction on the historical decomposition that simultaneously constrains all columns of $\boldsymbol{Q}$. Consequently, it is necessary to numerically approximate whether the identified set

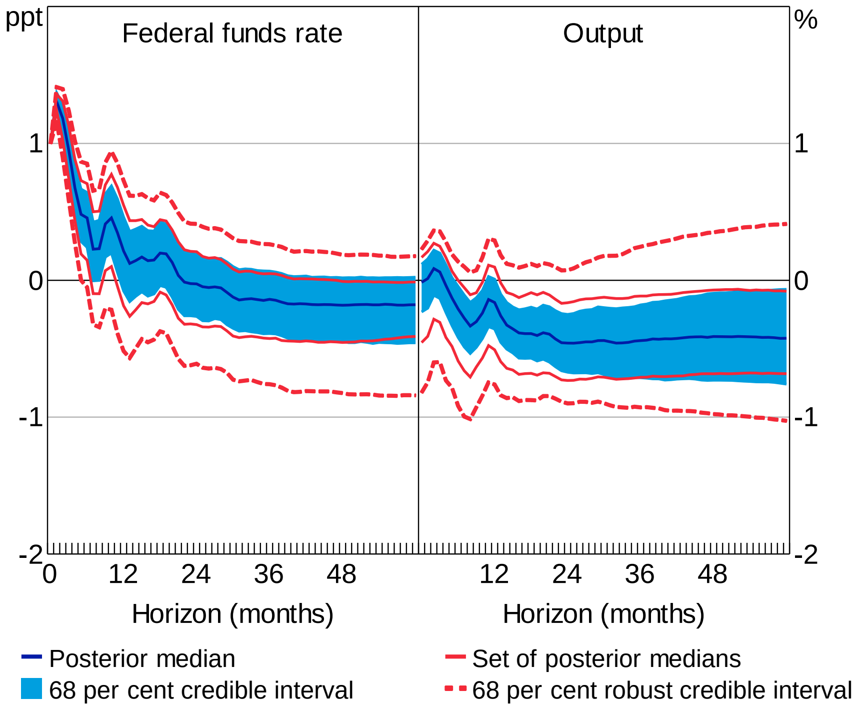
25 I approximate the bounds of the identified set at each draw of $\boldsymbol{\phi}$ by obtaining 10,000 draws of $\boldsymbol{Q}$ from a uniform distribution over $Q(\boldsymbol{\phi}|S,F)$, computing the impulse responses to a unit shock at each draw of $\boldsymbol{Q}$ and taking the minimum and maximum over the draws. I draw from the uniform distribution over $Q(\boldsymbol{\phi}|S,F)$ using the Gibbs sampler described in Read (forthcoming), which extends the Gibbs sampler proposed in Amir-Ahmadi and Drautzburg (2021) to additionally allow for zero restrictions.

26 For each horizon and variable of interest, I construct the 68 per cent robust credible interval by computing the 16th percentile of the posterior distribution of $\ell(\boldsymbol{\phi})$ and the 84th percentile of the posterior distribution of $u(\boldsymbol{\phi})$. This construction differs to the *shortest* robust credible interval that is proposed in Giacomini and Kitagawa (2021); computing the shortest credible interval requires searching over a grid of possible values, which can be computationally difficult when the identified set is sometimes unbounded.

27 Giacomini, Kitagawa and Read (2021a) discuss identification and inference under narrative restrictions. They propose extending the robust Bayesian approach to inference from Giacomini and Kitagawa (2021) to this setting to avoid undesirable features of the standard Bayesian approach to inference that arise when using narrative restrictions. Under narrative restrictions, the sign restrictions are functions of the data through the reduced-form VAR innovations that enter the restrictions. Consequently, the standard definition of an identified set does not apply; Giacomini *et al* (2021a) instead introduce the concept of a 'conditional' identified set, which is the identified set that would be obtained after conditioning on the data that directly enter the narrative restrictions. I leave this dependence implicit in the notation and refer to the identified set interchangeably with the conditional identified set.

for the impact response of the federal funds rate includes zero. Following the approach described in Section 4, I use 100,000 draws of $\boldsymbol{Q}$ from a uniform distribution over $Q(\boldsymbol{\phi}|\tilde{F})$ to check this.[28]

Figure 4: Impulse Responses to 100 Basis Point Shock – Restriction (3)



Notes: Results obtained under a combination of the identifying restrictions in Uhlig (2005), Antolín-Díaz and Rubio-Ramírez (2018) and Arias *et al* (2019).

Under Restriction (3), the identified set for the impact response of the federal funds rate excludes zero in 100 per cent of draws from the reduced-form posterior, which implies that the identified sets for the impulse responses to a 100 basis point shock are always bounded. Consequently, the sets of posterior means and all posterior quantiles for the impulse responses to a 100 basis point shock are bounded. The set of posterior medians excludes zero at most horizons and the 68 per cent robust credible intervals are substantially narrower than under Restriction (2) (Figure 4). Nevertheless, the robust credible intervals continue to include zero at all horizons and there remains substantial uncertainty about the output response to a 100 basis point shock.

Table 1 tabulates the posterior lower and upper probabilities that the decline in output is more extreme than a given threshold at selected horizons. Under Restriction (2), the posterior lower and upper probabilities that the output response is negative include both small values and values close to one at all horizons, which indicates that the data and identifying restrictions are fairly uninformative about the sign of the output response. In contrast, under Restriction (3), the posterior lower probability that the output response is negative at the two-year horizon is around 75 per cent and the posterior upper probability of this hypothesis is 100 per cent. The hypothesis that output declines following a positive monetary policy shock therefore receives reasonably high posterior

28 I approximate the identified set as being empty if I cannot obtain a draw of $\boldsymbol{Q}$ satisfying the identifying restrictions after 100,000 draws. Conditional on the identified set being nonempty, I approximate its bounds by obtaining 10,000 draws from a uniform distribution over $Q(\boldsymbol{\phi}|\boldsymbol{S},F)$ and computing the minimum and maximum impulse responses over these draws. The numerical methods used to obtain the results under these restrictions are computationally burdensome, so I base the results on 1,000 (rather than 10,000) draws of $\boldsymbol{\phi}$ such that the identified set is nonempty.

probability uniformly over the class of posteriors that are consistent with the identifying restrictions. Both sets of identifying restrictions effectively rule out relatively large declines in output following a 100 basis point shock; for example, under Restriction (3), the posterior lower probability that output declines by more than 1 per cent two years after the shock is zero and the posterior upper probability is around 5 per cent.

Table 1: Posterior Lower and Upper Probabilities that Decline in Output Exceeds Threshold Following 100 Basis Point Shock[(a)]

Horizon\threshold (%)	Lower probability				Upper probability			
	0	–0.25	–0.5	–1	0	–0.25	–0.5	–1
				Restriction (2)				
Impact	0.00	0.00	0.00	0.00	0.95	0.75	0.48	0.10
One year	0.13	0.01	0.00	0.00	0.99	0.84	0.48	0.06
Two years	0.27	0.11	0.03	0.00	1.00	1.00	0.93	0.07
Three years	0.23	0.11	0.04	0.00	1.00	0.99	0.88	0.11
Four years	0.23	0.12	0.05	0.00	1.00	0.98	0.80	0.15
				Restriction (3)				
Impact	0.00	0.00	0.00	0.00	0.95	0.73	0.45	0.04
One year	0.27	0.03	0.00	0.00	0.99	0.84	0.47	0.04
Two years	0.76	0.36	0.08	0.00	1.00	1.00	0.93	0.06
Three years	0.64	0.33	0.11	0.01	1.00	0.99	0.88	0.10
Four years	0.57	0.30	0.12	0.01	1.00	0.98	0.79	0.13

Note: (a) Posterior lower (upper) probability is the smallest (largest) posterior probability obtainable within the class of posteriors consistent with the identifying restrictions.

The existing literature contains a wide range of estimates for the output effects of a 100 basis point shock to the federal funds rate; for example, Ramey (2016) reports a range of existing estimates for the trough in the response of output under different samples, specifications and approaches to identification. These estimates range from as low as 0.6 per cent to as high as 5 per cent. The estimates tend to suggest that the trough in the response of output occurs around two years after the shock, which is consistent with the estimates obtained under Restriction (3). The results under Restriction (3) are broadly consistent with the effects of monetary policy lying towards the smaller end of the range of existing estimates.

6. Ruling Out Unboundedness Using Alternative Restrictions

In the context of estimating the effects of monetary policy, unboundedness of the impulse responses to a unit shock may arise when the identified set for the impact response of the federal funds rate to the monetary policy shock includes zero. A zero value for this impulse response may strike some researchers as implausible. Imposing sign, zero or narrative restrictions of the types considered above can indirectly rule this possibility out. This section discusses alternative restrictions that could potentially be used to rule out the possibility that the monetary policy shock has no impact effect on the federal funds rate. Although this discussion is framed in the context of estimating the effects of monetary policy, it also applies more generally to other settings.

6.1 Direct Bounds on Impulse Responses

One possibility is to directly restrict the impact response of the federal funds rate to the monetary policy shock so that it is greater than some (strictly positive) number (i.e. $\boldsymbol{e}'_{1,n}\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 \geq \lambda$, where $\lambda > 0$ is a specified scalar). However, it seems difficult to justify such restrictions on the basis of economic theory – what is the smallest plausible impact effect of a 'standard-deviation' monetary policy shock on the federal funds rate? Restrictions of this type could potentially be justified on the basis of prior estimates (e.g. from other SVARs or from estimated DSGE models), but these prior estimates may themselves be based on assumptions that lack credibility. Conversely, one could impose bounds on the responses of variables to a unit shock such that unbounded impulse responses are ruled out by assumption. However, it seems similarly difficult to come up with hard bounds on the responses of variables to a 100 basis point shock without these bounds being somewhat arbitrary. Moreover, in either case, when identified sets are unbounded in the absence of such restrictions, inferences may be highly sensitive to changes in the imposed bounds.

To illustrate, return to the bivariate example of Section 3 and consider imposing (in addition to the sign restrictions) the restriction that $\eta_{1,1,0} \geq \lambda$ for some $\lambda > 0$. When $\sigma_{21} \geq 0$ and $0 < \lambda \leq \sigma_{11}$, the identified set for $\tilde{\eta}_{2,1,0}$ is[29]

$$\tilde{\eta}_{2,1,0} \in \left[\frac{\sigma_{21}}{\sigma_{11}} - \frac{\sigma_{22}}{\lambda}\sqrt{\left(1 - \left(\frac{\lambda}{\sigma_{11}}\right)^2\right)}, 0\right].$$

The additional restriction therefore results in the identified set being bounded; in the absence of this restriction (or in the limit as λ converges to zero from above), the identified set is $(-\infty, 0]$. However, the lower bound of the identified set is sensitive to the choice of λ, particularly for small values of λ; the derivative of the lower bound tends to ∞ as λ approaches zero from above. Setting λ to some small positive number to rule out an unbounded identified set for $\tilde{\eta}_{2,1,0}$ will therefore yield an identified set that is highly sensitive to the exact choice of λ.

6.2 Bounds on the Forecast Error Variance Decomposition

Rather than directly restricting the impact effect of the monetary policy shock on the federal funds rate, one could instead consider restricting the one-step-ahead forecast error variance decomposition (FEVD) of the federal funds rate with respect to the monetary policy shock. This is the contribution of the monetary policy shock to the one-step-ahead forecast error variance (FEV) of the federal funds rate. For example, Volpicella (forthcoming) proposes imposing bounds on the FEVD, where the bounds are elicited from a range of estimated DSGE models.

Such restrictions may indirectly rule out the possibility that the monetary policy shock has no impact effect on the federal funds rate; intuitively, a strictly positive lower bound on the contribution of the monetary policy shock to the one-step-ahead FEV of the federal funds rate means that the impact

29 If $\lambda > \sigma_{11}$, the identified set is empty. See Appendix A.3 for details about this example.

effect of the shock itself must be strictly positive. More formally, the horizon-h FEVD of the ith variable with respect to the jth shock is

$$FEVD_{i,j,h}(\boldsymbol{\phi},\boldsymbol{Q}) = \frac{\sum_{l=0}^{h-1} \boldsymbol{c}_{il}'(\boldsymbol{\phi})\boldsymbol{q}_j\boldsymbol{q}_j\boldsymbol{c}_{il}(\boldsymbol{\phi})}{\sum_{l=0}^{h-1} \boldsymbol{c}_{il}'(\boldsymbol{\phi})\boldsymbol{c}_{il}(\boldsymbol{\phi})}.$$

The impact effect of the jth shock on the ith variable ($\boldsymbol{c}_{i0}'(\boldsymbol{\phi})\boldsymbol{q}_j = \boldsymbol{e}_{i,n}'\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_j$) is zero if and only if $FEVD_{i,j,0}(\boldsymbol{\phi},\boldsymbol{Q}) = 0$, so bounding $FEVD_{i,j,0}(\boldsymbol{\phi},\boldsymbol{Q})$ away from zero indirectly bounds the impact response away from zero. However, if the assumptions underlying the DSGE models that are used to elicit these bounds lack credibility, the derived bounds on the FEVDs will also lack credibility. As in the case where the normalising impulse response is directly bounded away from zero, the identified set obtained under some small lower bound on the FEVD will also be sensitive to the choice of this lower bound when the identified set is unbounded in the absence of this restriction (see Appendix A.3 for an analysis of this case in the context of the bivariate model).

7. Conclusion

In SVARs that are set-identified using sign and/or zero restrictions, the identified set for the impulse responses to a unit shock may be unbounded. This raises complications when conducting inference about these impulse responses, because particular inferential outputs may be unbounded. However, it may still be possible to draw useful inferences about impulse responses to a unit shock when the identified set is unbounded with positive posterior probability.

The empirical exercise in this paper demonstrates the importance of these issues. Under the identifying restrictions considered in Arias *et al* (2019), the identified set for the impulse response to a 100 basis point monetary policy shock appears to be unbounded at all horizons and for all values of the reduced-form parameters. The identifying restrictions are therefore extremely uninformative about the magnitude of these impulse responses, standard Bayesian inference is misleading about the information contained in the data given the identifying restrictions, and posterior inferences are highly sensitive to the choice of conditional prior for the orthonormal matrix. After adding the sign restrictions on impulse responses from Uhlig (2005), the identified set is bounded with high posterior probability. Additionally adding the narrative restrictions from Antolín-Díaz and Rubio-Ramírez (2018) yields a bounded identified set in 100 per cent of draws from the reduced-form posterior. The results under the latter two sets of restrictions are broadly consistent with the effects of US monetary policy lying towards the smaller end of the range of existing estimates.

Appendix A: Derivations for Bivariate SVAR

A.1 Sign Restrictions on Impulse Responses

This appendix derives the identified sets for the impulse responses to a unit shock under the sign restrictions on impulse responses presented in Section 3.

In the absence of any identifying restrictions, the identified set for $\boldsymbol{A}_0^{-1}$ (the matrix of impact impulse responses) is

$$\boldsymbol{A}_0^{-1} \in \left\{\begin{bmatrix} \sigma_{11}\cos\theta & -\sigma_{11}\sin\theta \\ \sigma_{21}\cos\theta + \sigma_{22}\sin\theta & \sigma_{22}\cos\theta - \sigma_{21}\sin\theta \end{bmatrix}\right\} \cup \left\{\begin{bmatrix} \sigma_{11}\cos\theta & \sigma_{11}\sin\theta \\ \sigma_{21}\cos\theta + \sigma_{22}\sin\theta & \sigma_{21}\sin\theta - \sigma_{22}\cos\theta \end{bmatrix}\right\},$$

and the identified set for $\boldsymbol{A}_0$ is

$$\boldsymbol{A}_0 \in \left\{\frac{1}{\sigma_{11}\sigma_{22}}\begin{bmatrix} \sigma_{22}\cos\theta - \sigma_{21}\sin\theta & \sigma_{11}\sin\theta \\ -\sigma_{21}\cos\theta - \sigma_{22}\sin\theta & \sigma_{11}\cos\theta \end{bmatrix}\right\} \cup \left\{\frac{1}{\sigma_{11}\sigma_{22}}\begin{bmatrix} \sigma_{22}\cos\theta - \sigma_{21}\sin\theta & \sigma_{11}\sin\theta \\ \sigma_{21}\cos\theta + \sigma_{22}\sin\theta & -\sigma_{11}\cos\theta \end{bmatrix}\right\}.$$

The impact response of the second variable to a shock that raises the first variable by one unit is

$$\tilde{\eta}_{2,1,0} = \frac{\eta_{2,1,0}}{\eta_{1,1,0}} = \frac{\sigma_{21}\cos\theta + \sigma_{22}\sin\theta}{\sigma_{11}\cos\theta} = \frac{\sigma_{21}}{\sigma_{11}} + \frac{\sigma_{22}}{\sigma_{11}}\tan\theta.$$

Under the sign restrictions on impulse responses described in Section 3 (including the sign normalisation), the parameter θ is restricted to lie within the following set:

$$\theta \in \{\theta: \sigma_{11}\cos\theta \geq 0, \sigma_{21}\cos\theta \leq -\sigma_{22}\sin\theta, \sigma_{22}\cos\theta \geq \sigma_{21}\sin\theta\}$$

$$\cup \{\theta: \sigma_{11}\cos\theta \geq 0, \sigma_{21}\cos\theta \leq -\sigma_{22}\sin\theta, \sigma_{22}\cos\theta \geq \sigma_{21}\sin\theta, -\sigma_{11}\cos\theta \geq 0\}.$$

There are two cases to consider depending on the sign of σ_{21}. If $\sigma_{21} < 0$, the second set is empty. The first set is equivalent to

$$\left\{\theta: \cos\theta > 0, \tan\theta \leq -\frac{\sigma_{21}}{\sigma_{22}}, \tan\theta \geq \frac{\sigma_{22}}{\sigma_{21}}\right\}.$$

This set of inequalities implies that the identified set for θ is

$$\theta \in \left[\arctan\left(\frac{\sigma_{22}}{\sigma_{21}}\right), \arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right].$$

The identified set for the impact response of the first variable to the first shock, $\eta_{11} = \sigma_{11}\cos\theta$, is then

$$\eta_{1,1,0} \in \left[\sigma_{11}\cos\left(\arctan\left(\min\left\{\frac{\sigma_{22}}{\sigma_{21}}, \frac{\sigma_{21}}{\sigma_{22}}\right\}\right)\right), \sigma_{11}\right].$$

The identified set for this impulse response is bounded away from zero. In this case, the identified set for $\tilde{\eta}_{2,1,0}$ is

$$\tilde{\eta}_{2,1,0} \in \left[\frac{\sigma_{21}}{\sigma_{11}} + \frac{\sigma_{22}^2}{\sigma_{11}\sigma_{21}}, 0\right],$$

which is bounded.

Similarly, if $\sigma_{21} > 0$, θ is restricted to lie in the set

$$\theta \in \left\{\theta: \cos\theta > 0, \tan\theta \leq -\frac{\sigma_{21}}{\sigma_{22}}, \tan\theta \leq \frac{\sigma_{22}}{\sigma_{21}}\right\} \cup \left\{-\frac{\pi}{2}\right\}.$$

The second inequality implies that $\tan\theta \leq 0$, so the last inequality never binds. The identified set for θ is therefore

$$\theta \in \left[-\frac{\pi}{2}, \arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right],$$

and the identified set for $\eta_{1,1,0}$ is

$$\eta_{1,1,0} \in \left[0, \sigma_{11}\cos\left(\arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right)\right].$$

If $\sigma_{21} = 0$, θ is restricted to the set

$$\theta \in \{\theta: \cos\theta \geq 0, 0 \leq -\sigma_{22}\sin\theta\} \cup \{\theta: 0 \leq -\sigma_{22}\sin\theta, \cos\theta \geq 0, -\sigma_{11}\cos\theta \geq 0\}.$$

The first set implies $\theta \in [-\pi/2, 0]$ and the second implies $\theta = -\pi/2$, so $\eta_{1,1,0} \in [0, \sigma_{11}]$. The expression for the identified set for $\eta_{1,1,0}$ when $\sigma_{21} > 0$ therefore also applies when $\sigma_{21} = 0$. $\tan\theta \rightarrow -\infty$ as θ approaches $-\pi/2$ from above. $\tan\theta$ is strictly increasing over the identified set for θ, so the upper bound for the identified set for $\tilde{\eta}_{2,1,0}$ is obtained by evaluating $\tilde{\eta}_{2,1,0}$ at the upper bound of the identified set for θ. Consequently, $\tilde{\eta}_{2,1,0} \in (-\infty, 0]$.

A.2 Sign Restrictions on Impulse Responses to Multiple Shocks

If we additionally impose the sign restrictions that $\eta_{1,2,0} \equiv \mathbf{e}_{1,2}'\mathbf{A}_0^{-1}\mathbf{e}_{2,2} \geq 0$ and $\eta_{2,2,0} \equiv \mathbf{e}_{2,2}'\mathbf{A}_0^{-1}\mathbf{e}_{2,2} \geq 0$, the parameter θ is restricted to lie within the following set:

$$\theta \in \{\theta: \sigma_{11}\cos\theta \geq 0, \sigma_{21}\cos\theta \leq -\sigma_{22}\sin\theta, \sigma_{22}\cos\theta \geq \sigma_{21}\sin\theta, -\sigma_{11}\sin\theta \geq 0\}$$

$$\cup \left\{\begin{matrix}\theta: \sigma_{11}\cos\theta \geq 0, \sigma_{21}\cos\theta \leq -\sigma_{22}\sin\theta, \sigma_{22}\cos\theta \geq \sigma_{21}\sin\theta, -\sigma_{11}\cos\theta \geq 0, \\ \sigma_{21}\sin\theta \geq \sigma_{22}\cos\theta, \sigma_{11}\sin\theta \geq 0\end{matrix}\right\}.$$

Using working similar that in Appendix A.1, the identified sets for θ, $\eta_{1,1,0}$ and $\tilde{\eta}_{2,1,0}$ are given by:

$$\theta \in \begin{cases} \left[\arctan\left(\frac{\sigma_{22}}{\sigma_{21}}\right), 0\right] & \text{if } \sigma_{21} < 0 \\ \left[-\frac{\pi}{2}, \arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right] & \text{if } \sigma_{21} \geq 0 \end{cases}$$

$$\eta_{1,1,0} \in \begin{cases} \left[\sigma_{11}\cos\left(\arctan\left(\frac{\sigma_{22}}{\sigma_{21}}\right)\right), \sigma_{11}\right] & \text{if } \sigma_{21} < 0 \\ \left[0, \sigma_{11}\cos\left(\arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right)\right] & \text{if } \sigma_{21} \geq 0 \end{cases}$$

$$\tilde{\eta}_{2,1,0} \in \begin{cases} \left[\frac{\sigma_{21}}{\sigma_{11}} + \frac{\sigma_{22}^2}{\sigma_{11}\sigma_{21}}, \frac{\sigma_{21}}{\sigma_{11}}\right] & \text{if } \sigma_{21} < 0 \\ (-\infty, 0] & \text{if } \sigma_{21} \geq 0 \end{cases}$$

As in the case where there are sign restrictions on the impulse responses to the first shock only, the identified set for $\eta_{1,1,0}$ includes zero when $\sigma_{21} \geq 0$ and the identified set for $\tilde{\eta}_{2,1,0}$ is unbounded. In the case where $\sigma_{21} \geq 0$, the additional sign restrictions serve to tighten the identified set (the upper bound is now strictly less than zero).

A.3 Magnitude Restrictions

In addition to the sign restrictions considered in the previous section, consider the restriction that $\eta_{1,1,0} \geq \lambda$ for some $\lambda > 0$. Under this set of restrictions, θ is restricted to lie within the set:

$$\theta \in \{\theta\colon \sigma_{11}\cos\theta \geq \lambda, \sigma_{21}\cos\theta \leq -\sigma_{22}\sin\theta, \sigma_{22}\cos\theta \geq \sigma_{21}\sin\theta\}$$

$$\cup \{\theta\colon \sigma_{11}\cos\theta \geq \lambda, \sigma_{21}\cos\theta \leq -\sigma_{22}\sin\theta, \sigma_{22}\cos\theta \geq \sigma_{21}\sin\theta, -\sigma_{11}\cos\theta \geq 0\}.$$

The second set is always empty, since $\sigma_{11}\cos\theta \geq \lambda$ and $-\sigma_{11}\cos\theta \geq 0$ cannot hold simultaneously when $\lambda > 0$. The identified set for θ is empty if $\lambda > \sigma_{11}$, since $\cos\theta \leq 1$ for all θ.

If $\sigma_{21} \geq 0$, the first set is equivalent to

$$\theta \in \left\{\theta\colon \cos\theta \geq \frac{\lambda}{\sigma_{11}}, \tan\theta \leq -\frac{\sigma_{21}}{\sigma_{22}}, \tan\theta \leq \frac{\sigma_{22}}{\sigma_{21}}\right\}.$$

The last inequality never binds and the identified set for θ is

$$\theta \in \left[-\arccos\left(\frac{\lambda}{\sigma_{11}}\right), \arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right].$$

The identified set for $\tilde{\eta}_{2,1,0}$ is therefore

$$\tilde{\eta}_{2,1,0} \in \left[\frac{\sigma_{21}}{\sigma_{11}} + \frac{\sigma_{22}}{\sigma_{11}}\tan\left(-\arccos\left(\frac{\lambda}{\sigma_{11}}\right)\right), 0\right].$$

The lower bound of this identified set, $\ell(\boldsymbol{\phi}, \lambda)$, can be expressed as

$$\ell(\boldsymbol{\phi}, \lambda) = \frac{\sigma_{21}}{\sigma_{11}} - \frac{\sigma_{22}}{\lambda}\sqrt{\left(1 - \left(\frac{\lambda}{\sigma_{11}}\right)^2\right)},$$

which converges to $-\infty$ as λ approaches zero from above. The derivative of $\ell(\boldsymbol{\phi},\lambda)$ with respect to λ is

$$\frac{\partial \ell(\boldsymbol{\phi},\lambda)}{\partial \lambda}=\frac{\left(1-\left(\frac{\lambda}{\sigma_{11}}\right)^2\right)^{-\frac{1}{2}}}{\lambda^2}.$$

In the limit as λ approaches zero from above, this derivative approaches ∞, which implies that the lower bound is extremely sensitive to small changes in λ when λ is close to zero.

A.4 Bounds on the FEVD

The FEV of of y_{1t} is σ_{11}^2 and the contribution of ε_{1t} to the FEV of y_{1t} is $\sigma_{11}^2\cos^2\theta$. The FEVD of y_{1t} with respect to ε_{1t}, $FEVD_{\varepsilon_{1t}}^{y_{1t}}$, is therefore $\cos^2\theta$. Consider imposing the restriction that $FEVD_{\varepsilon_{1t}}^{y_{1t}}\geq\kappa$ for some $0<\kappa\leq 1$ in addition to the sign restrictions considered in Section 3 and Appendix A.1. Under this set of restrictions, θ is restricted to lie within the following set:

$$\theta\in\{\theta:\sigma_{11}\cos\theta\geq 0,\sigma_{21}\cos\theta\leq-\sigma_{22}\sin\theta,\sigma_{22}\cos\theta\geq\sigma_{21}\sin\theta,\cos^2\theta\geq\kappa\}$$

$$\cup\{\theta:\sigma_{11}\cos\theta\geq 0,\sigma_{21}\cos\theta\leq-\sigma_{22}\sin\theta,\sigma_{22}\cos\theta\geq\sigma_{21}\sin\theta,-\sigma_{11}\cos\theta\geq 0,\cos^2\theta\geq\kappa\}.$$

When $\sigma_{21}\geq 0$, the first set is equivalent to

$$\theta\in\left\{\theta:\ \cos\theta>0,\tan\theta\leq-\frac{\sigma_{21}}{\sigma_{22}},\tan\theta\leq\frac{\sigma_{22}}{\sigma_{21}},-\arccos\sqrt{\kappa}\leq\theta\leq\arccos\sqrt{\kappa}\right\}.$$

The inequalities $\tan\theta\leq\sigma_{22}/\sigma_{21}$ and $\theta\leq\arccos\sqrt{\kappa}$ never bind and the identified set for θ is

$$\theta\in\left[-\arccos\sqrt{\kappa},\arctan\left(-\frac{\sigma_{21}}{\sigma_{22}}\right)\right].$$

The identified set for $\tilde{\eta}_{2,1,0}$ is therefore

$$\tilde{\eta}_{2,1,0}\in\left[\frac{\sigma_{21}}{\sigma_{11}}+\frac{\sigma_{22}}{\sigma_{11}}\tan\left(-\arccos\left(\sqrt{\kappa}\right)\right),0\right].$$

The lower bound of this identified set, $\ell(\boldsymbol{\phi},\kappa)$, can be expressed as

$$l(\boldsymbol{\phi},\kappa)=\frac{\sigma_{21}}{\sigma_{11}}-\frac{\sigma_{22}}{\sigma_{11}}\frac{\sqrt{1-\kappa}}{\sqrt{\kappa}}.$$

The lower bound converges to $-\infty$ as κ approaches zero from above. The derivative of $\ell(\boldsymbol{\phi},\kappa)$ with respect to κ is

$$\frac{\partial \ell(\boldsymbol{\phi},\kappa)}{\partial \kappa}=\frac{1}{2}\frac{\sigma_{22}}{\sigma_{11}}\kappa^{-\frac{3}{2}}(1-\kappa)^{-\frac{1}{2}}.$$

In the limit as κ approaches zero from above, this derivative approaches ∞, which implies that the lower bound is extremely sensitive to small changes in κ when κ is close to zero.

To summarise, under the additional restriction on the FEVD, the identified set is bounded; in the absence of this restriction (or as κ converges to zero from above), the identified set is $(-\infty, 0]$. However, as in the case where the normalising impulse response is directly bounded away from zero, the lower bound of the identified set is sensitive to the choice of κ, particularly for small values of κ; the derivative of the lower bound tends to ∞ as κ approaches zero from above. Setting κ to some small positive number to rule out an unbounded identified set for $\tilde{\eta}_{2,1,0}$ will therefore yield an identified set that is highly sensitive to the choice of κ.

Appendix B: Proofs of Propositions

Proof of Proposition 4.1. Assume $\eta_{1,1,0}(\boldsymbol{\phi}|S,F)$ does not include zero, so $\boldsymbol{e}'_{1,n}\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 > 0$ for any $\boldsymbol{Q} \in \mathcal{Q}(\boldsymbol{\phi}|S,F)$. There exists $\delta > 0$ such that $\boldsymbol{e}'_{1,n}\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 > \delta$ for all $\boldsymbol{Q} \in \mathcal{Q}(\boldsymbol{\phi}|S,F)$. Given that the impulse response horizon h is fixed and finite, $|\eta_{i,j,h}(\boldsymbol{\phi},\boldsymbol{Q})| < \infty$ for all $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ and $\boldsymbol{Q} \in \mathcal{Q}(\boldsymbol{\phi}|S,F)$.[30] There thus exists $\kappa < \infty$ such that $|\eta_{i,1,h}(\boldsymbol{\phi},\boldsymbol{Q})| < \kappa$ for all $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ and $\boldsymbol{Q} \in \mathcal{Q}(\boldsymbol{\phi}|S,F)$. It follows that $|\tilde{\eta}_{i,1,h}(\boldsymbol{\phi},\boldsymbol{Q})| < \frac{\kappa}{\delta} < \infty$ for all $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ and $\boldsymbol{Q} \in \mathcal{Q}(\boldsymbol{\phi}|S,F)$, so $\tilde{\eta}_{i,1,h}(\boldsymbol{\phi}|S,F)$ must be bounded. □

Proof of Proposition 4.2. Assume that the sign restrictions are ordered such that the first row of $S(\boldsymbol{\phi})$ is $\boldsymbol{e}'_{1,n}\boldsymbol{\Sigma}_{tr}$. Since $\boldsymbol{\Sigma}_{tr}$ is lower triangular, the condition $\boldsymbol{e}'_{1,n}\boldsymbol{\Sigma}_{tr}\boldsymbol{q}_1 = 0$ is satisfied only for values of $\boldsymbol{q}_1$ such that $\boldsymbol{q}_1 = (0,\boldsymbol{q}'_{1,2:n})'$, where $\boldsymbol{q}_{1,2:n}$ is an $(n-1)$-dimensional vector.[31] For such a value of $\boldsymbol{q}_1$, the entries in the first columns of $F(\boldsymbol{\phi})$ and $S(\boldsymbol{\phi})$ do not enter the equality restrictions in $F(\boldsymbol{\phi})\boldsymbol{q}_1 = \boldsymbol{0}_{r\times 1}$ and the last $s-1$ inequalities in $S(\boldsymbol{\phi})\boldsymbol{q}_1 \geq \boldsymbol{0}_{s\times 1}$, respectively. Let $\ddot{F}(\boldsymbol{\phi})$ represent the matrix of coefficients in the zero restrictions after dropping the first column and let $\ddot{S}(\boldsymbol{\phi})$ represent the matrix of coefficients in the sign restrictions after dropping the first row and column. According to Proposition 3.1 of Read (forthcoming), the system of sign and zero restrictions in $\mathbb{R}^{n-1}$, $\ddot{F}(\boldsymbol{\phi})\boldsymbol{q}_{1,2:n} = \boldsymbol{0}_{f\times 1}$ and $\ddot{S}(\boldsymbol{\phi})\boldsymbol{q}_{1,2:n} \geq \boldsymbol{0}_{(s-1)\times 1}$, can be expressed as an equivalent system of sign restrictions in $\mathbb{R}^{n-f-1}$. Let $\breve{S}(\boldsymbol{\phi})\breve{\boldsymbol{q}} = \boldsymbol{0}_{(s-1)\times 1}$ represent the transformed system of sign restrictions, where $\breve{\boldsymbol{q}} \in \mathbb{R}^{n-f-1}$ and $\breve{S}(\boldsymbol{\phi})$ is obtained from $\ddot{S}(\boldsymbol{\phi})$ using the transformation described in Read (forthcoming). Corollary 3.1 of Read (forthcoming) implies that the identified set for $\eta_{1,1,0}$ will include zero if and only if there exists $\breve{\boldsymbol{q}}$ satisfying $\breve{S}(\boldsymbol{\phi})\breve{\boldsymbol{q}} \geq \boldsymbol{0}_{(s-1)\times 1}$. In what follows, I show that such a vector always exists under the assumptions of the proposition.

Consider the case where $\mathrm{rank}\left(\breve{S}(\boldsymbol{\phi})\right) = s-1$. By Gordan's Theorem (e.g. Mangasarian 1969; Border 2020), either $\breve{S}(\boldsymbol{\phi})\boldsymbol{x} > \boldsymbol{0}_{(s-1)\times 1}$ for some $\boldsymbol{x} \in \mathbb{R}^{n-f-1}$ or $\breve{S}(\boldsymbol{\phi})'\boldsymbol{y} = \boldsymbol{0}_{(n-r-1)\times 1}$ for some $\boldsymbol{y} \neq \boldsymbol{0}_{(s-1)\times 1}$. Since $\breve{S}(\boldsymbol{\phi})$ has full rank, there cannot exist a $\boldsymbol{y} \neq \boldsymbol{0}_{(s-1)\times 1}$ such that $\breve{S}(\boldsymbol{\phi})'\boldsymbol{y} = \boldsymbol{0}_{(n-f-1)\times 1}$, so there must exist $\breve{\boldsymbol{q}}$ satisfying $\breve{S}(\boldsymbol{\phi})\breve{\boldsymbol{q}} > \boldsymbol{0}_{(s-1)\times 1}$. Any $\breve{\boldsymbol{q}}$ satisfying $\breve{S}(\boldsymbol{\phi})\breve{\boldsymbol{q}} > \boldsymbol{0}_{(s-1)\times 1}$ also satisfies $\breve{S}(\boldsymbol{\phi})\breve{\boldsymbol{q}} \geq \boldsymbol{0}_{(s-1)\times 1}$. Next, consider the case where $\mathrm{rank}\left(\breve{S}(\boldsymbol{\phi})\right) < s-1$ and let $N(\breve{S}(\boldsymbol{\phi}))$ represent an orthonormal basis for the null space of $\breve{S}(\boldsymbol{\phi})$. By the rank-nullity theorem, $N(\breve{S}(\boldsymbol{\phi}))$ has dimension $(n-f-1) - \mathrm{rank}\left(\breve{S}(\boldsymbol{\phi})\right)$. Since $\mathrm{rank}\left(\breve{S}(\boldsymbol{\phi})\right) < s-1$ and $s+f \leq n$, $N(\breve{S}(\boldsymbol{\phi}))$ will have dimension strictly greater than one. Thus, when $\mathrm{rank}\left(\breve{S}(\boldsymbol{\phi})\right) < s-1$, it is always possible to construct a unit-length vector satisfying $\breve{S}(\boldsymbol{\phi})\breve{\boldsymbol{q}} = \boldsymbol{0}_{(s-1)\times 1}$ by taking any column of $N(\breve{S}(\boldsymbol{\phi}))$. Such a vector clearly satisfies $\breve{S}(\boldsymbol{\phi})\breve{\boldsymbol{q}} \geq \boldsymbol{0}_{(s-1)\times 1}$. □

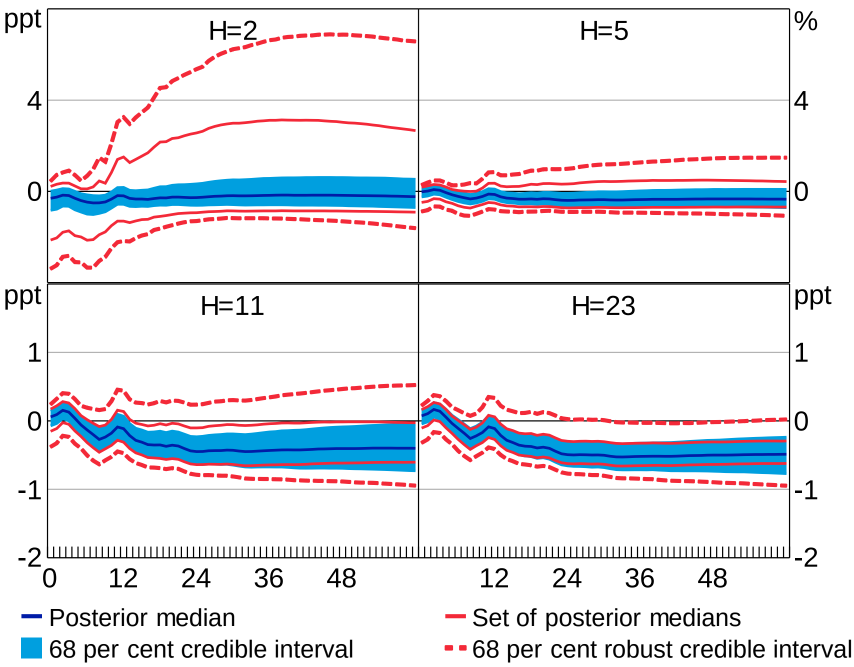
30 Allowing for arbitrarily large impulse-response horizons h would require restricting the support of the reduced-form parameter space $\boldsymbol{\Phi}$ such that the infinite-order vector moving average representation of the VAR exists; this will be the case if the eigenvalues of the companion matrix lie inside the unit circle (e.g. Hamilton 1994; Kilian and Lütkepohl 2017). By avoiding this assumption I allow for mildly explosive processes.

31 This assumes that the $(1,1)$ element of $\boldsymbol{\Sigma}_{tr}$ is nonzero, which is guaranteed so long as $\boldsymbol{\Sigma}$ is nonsingular.

Appendix C: Additional Empirical Results

Figure 5 presents additional results obtained under Restriction (2) when the horizon over which the sign restrictions are imposed (H) is varied. For $H = 2$, the identified set for $\eta_{1,1,0}$ includes zero in only 1.2 per cent of draws from the posterior. However, the set of posterior medians and 68 per cent robust credible intervals for the output response are extremely wide. Increasing H reduces the proportion of draws where the identified set for $\eta_{1,1,0}$ includes zero: for $H = 5$, the identified set includes zero in 0.6 per cent of draws; and for $H = 11$ or 23, there are no draws where the identified set includes zero. Increasing the number of sign restrictions also appreciably tightens the set of posterior medians and robust credible intervals.

Figure 5: Impulse Responses to 100 Basis Point Shock – Alternative Horizons



Notes: Results obtained under a combination of the identifying restrictions in Uhlig (2005) and Arias *et al* (2019). H is the horizon over which the impulse responses are restricted.

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