Take the Short Route*

How to repay and restructure sovereign debt with multiple maturities

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Abstract

We address the question of whether and how a sovereign should reduce its external indebtedness when default is a significant possibility, with a particular focus on whether a sovereign should buy back or dilute existing long-term sovereign bonds. Our main finding is that when reduction of debt is optimal, the sovereign should remain passive in the long-term bond market during the deleveraging process, retiring long-term bonds as they mature but never actively issuing or buying back these bonds. The only active margin is the short-term bond market, which involves partial roll over of such debt. Any active maturity management, as will typically be required to address rollover crisis risk, will be delayed until the end of the deleveraging process. We also show that there exist a set of Pareto improving debt restructurings in which maturities are shortened; however, these cannot be implemented by trading in competitive secondary markets.

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1 Introduction

Short-term debt is often cast as the villain in sovereign debt crises, exposing fiscal budgets to sharp swings in interest rates and raising the vulnerability to a rollover crisis. Nevertheless, when faced with increased spreads on their bonds, sovereigns tend to lower their debt issuances while tilting the composition of new bonds toward shorter maturities.[1] This favoritism towards short-term debt during periods of crisis is somewhat puzzling, as there is wide agreement that short-term debt, if anything, introduces more risk by leaving the country vulnerable to rollover problems. Understanding debt dynamics – and associated equilibrium prices – in a crisis environment is of particular importance given the sovereign debt crisis in Europe. Many peripheral European countries are currently paying a significant premium over German debt on large quantities of sovereign bonds. The respective governments are contemplating fiscal paths that lead to lower debt-to-GDP ratios, and correspondingly a lower risk of default and associated spread. However, in a world of limited commitment, fiscal trajectories must be time consistent, and it is an open question whether the vulnerability to default provides sufficient incentive to deleverage and what role – if any – maturity plays.

In this paper, we introduce an environment which captures important elements of sovereign debt markets. In particular, the environment features the risk of default; an incentive to deleverage due to this risk of default; an inability to commit to fiscal trajectories; a dynamic choice of maturity; and equilibrium bond prices that reflect and constrain the government's debt decisions. Given this framework, our goal is to clearly and completely analyze the interaction of maturity choice, equilibrium prices, and the dynamic incentives to deleverage under the threat of default.

A primary contribution of the paper is to provide a transparent analysis of how a sovereign's lack of commitment – to repayment as well as to future debt trajectories – plays out in equilibrium. We solve for the government's equilibrium budget set and explore how it responds to the maturity of sovereign bonds. A major result is that the government's equilibrium budget set is maximized by not actively issuing or repurchasing long-term bonds while deleveraging; and any lengthening of maturity, which may be required to mitigate rollover risk, will be postponed until the completion of the deleveraging process. In particular, a strategy that relies on issuing or repurchasing long-term bonds during deleveraging shrinks

[1] These facts have been documented for the emerging market debt crises of the 1990s and 2000s. Broner et al. (2013) document that emerging markets reduce total debt issuances when spreads increase, but the reduction is particularly pronounced for bonds with maturity greater than 3 years, sharply reducing the average maturity of new issuances. Similarly, Arellano and Ramanarayanan (2012) document that during crisis periods for four emerging market economies, the average maturity of new debt shortens. Perez (2013) documents in a large sample of emerging markets that debt issuance drops when spreads are high, and the maturity profile of debt shortens considerably.

the government budget set. In this precise sense, using long-term bonds is more expensive, even though all bonds are priced in an actuarially fair manner.

In the model, an infinitely-lived sovereign borrows in the form of non-contingent bonds of differing maturity from global financial markets. When the sovereign is highly indebted, a risk of default arises from the sovereign's inability to commit to future payments coupled with the presence of shocks to the sovereign's costs of default. The sovereign, however, can mitigate the resulting default risk by reducing its outstanding stock of external debt.

In equilibrium, bond prices reflect the probability of default through the life-time of the bond. Short-term bond prices thus reflect the next period's probability of default, while long-term bond prices incorporate the equilibrium expectations of default in subsequent periods as well. As a result, the price of the long-term bond depends on the equilibrium speed at which the sovereign will reduce its debt. When auctioning long-term bonds, the sovereign would like to promise bond holders a quick path to lower debt levels in order to generate a high price for its bonds. However, as there is no mechanism to enforce such a promise, a time-consistency problem arises. Short-term bond prices are not sensitive to expectations of future fiscal trajectories, but rather reflect only the probability of default next period. Because this probability depends on the amount of debt outstanding next period, which is known at the time of issuance, the above time-consistency problem does not arise when auctioning short-term bonds.

Our first main result leverages this distinction to argue that, in equilibrium, a deleveraging strategy that relies only on issuances of short-term bonds is optimal. That is, an optimal deleveraging policy is one where the sovereign services interest payments of existing long-term bonds, pays off any maturing bonds, and all new issuances consist of short-term bonds only. However, this result does not, by itself, rule out the possibility that an alternative strategy that actively uses long-term bonds may be optimal as well. Our second main result establishes the sub-optimality of auctioning or repurchasing long-term bonds. This implication relies crucially on properties of equilibrium bond prices, and how they respond to maturity choice.

In particular, we show that the maturity composition affects equilibrium prices by altering the speed of develeraging through a "price effect." To see this price effect most transparently, consider a sovereign that will repay unless a shock is realized, in which case it is almost indifferent between repayment and default, but slightly prefers the latter. Before the shock is realized, the sovereign must pay a premium on any new bonds it issues, as bond holders need to be compensated for the possibility of default. However, the sovereign's near indifference implies that there is no associated benefit in exchange for this premium. In fact, the sovereign would strictly prefer to be able to commit to repay. In equilibrium, the only way to achieve this commitment is through a reduction in debt levels. Hence, market prices induce the sovereign to save. Note that this price effect is only relevant if the sovereign is issuing new bonds as it deleverages. This is the case if a large amount of debt is short term and must be rolled over at frequent intervals. The default premium in short-term bond prices is akin to a variable cost that must paid each period until default risk is reduced or eliminated. Long-term bonds also embed a default premium at the time of issuance, but from the perspective of later periods, this premium is a sunk cost. This implies that the shorter the average maturity of bonds, the faster the sovereign deleverages.

The fact that maturity affects the incentives to save raises the question of whether the government has an incentive to actively adjust the maturity of its outstanding debt. From the above discussion, any trade that attempts to lengthen the maturity of the outstanding debt by issuing more long-term bonds reduces the incentive to save. Therefore, the sale of long-term bonds drives down their price. An opposite trade, one that attempts to decrease the maturity by buying back long-term bonds, increases the incentive to save. Thus repurchases of long-term bonds drive up their price. We show that these adverse price movements of active trading shrink the budget set of the government. This is our second main result.

By slightly altering the benchmark model (which features a unique equilibrium), we show that our results are robust to the presence of coordination failures and rollover crises. That is, during the deleveraging process, it remains optimal to issue only short-term bonds even in this richer model. Importantly, active maturity management, with the goal of reducing the risk of coordination failures, should be delayed until the end of the deleveraging period.

Finally, we show that the outcome of the competitive equilibrium is not efficient. If the sovereign and its creditors could efficiently restructure debt, the outcome would be to *shorten* the maturity of the outstanding debt in order to provide the best incentives to deleverage quickly. However, this shortening of maturity *cannot* be implemented in equilibrium because long-term bond holders have an incentive to hold out and reap a capital gain from the increased speed of deleveraging associated with shorter maturities. Any lengthening of maturity to mitigate rollover risk should occur at the end of the deleveraging process and can be implemented at equilibrium prices.

Related Literature

Our paper relates to a large literature on maturity choice, both in corporate finance and macroeconomics. We review key strands of analysis here, highlighting how our contribution differs from and complements the existing literature. For expositional clarity, many of our modeling choices are designed to isolate our mechanism from the more familiar economic forces regarding maturity choice that have been established in the literature.

In their seminal paper on optimal fiscal policy, Lucas and Stokey (1983) discuss at length how maturity choice is a useful tool to provide incentives to a government that lacks commitment. Our model emphasizes default risk, something absent from their work. Moreover, the government in Lucas and Stokey (1983) has an incentive to manipulate the risk-free real interest rate to alter the value of outstanding long-term bonds, something ruled out by our small open economy framework. However, our model government can induce a capital loss on long-term bondholders by altering fiscal policy and raising the likelihood of default. A major theme of our analysis is how this affects bond prices and equilibrium maturity choice. In particular, we explore why the government does not issue additional long-term bonds, even though this would induce a capital loss on legacy bondholders.

Our model environment is closely related to the principal-agent framework analyzed by Hopenhayn and Werning (2008), who study the financing of an investment project in a long-term relationship between a lender and an entrepreneur. In particular, Hopenhayn and Werning (2008) introduce unobservable outside option shocks to the entrepreneur, and show that the efficient long-term contract features equilibrium default and can be implemented with a sequence of non-contingent (but defaultable) short-term debt contracts, a result that is closely related to our efficient restructuring analysis of Section 5. Our analysis modifies this environment to study a different set of questions. Rather than studying the efficient contract in a principal-agent problem, we consider the competitive equilibria that arises in a market consisting of noncontingent (but defaultable) bonds of various maturities. Our results highlight that the use of short-term debt is optimal even when the government has legacy long-term bonds (an initial condition that makes the resulting equilibrium allocation inefficient). In addition, we identify the sub-optimality of issuing or repurchasing long-term bonds in equilibrium, as well as the perverse incentives that long-term bonds induce for future governments to reduce their debt. Finally, we consider equilibrium maturity choice when short-term debt makes the sovereign vulnerable to coordination failures among creditors.

There is a corporate finance literature on default building on the canonical model of Leland (1994b). This literature typically focuses on the optimal default decision given a constant capital structure (which, depending on the exercise, may or may not be chosen optimally in the initial period). In contrast, our analysis emphasizes that the level and maturity of outstanding debt is an endogenous variable that may vary over time. In fact, these dynamics are the main focus.

The corporate finance and banking literature frequently builds on the fact that bankruptcy involves partially liquidation of an asset. This influences maturity choice in a variety of ways. The classic paper by Calomiris and Kahn (1991) demonstrates how short-term debt and the threat of liquidation can be used to discipline a manger (see also Diamond and

Rajan, 2001). In the international context, Jeanne (2009) notes that the threat to withdraw liquid capital from an economy may provide a government with incentives to respect property rights and enforce contracts. The fact that existing bondholders hold a claim on liquidated assets also makes them vulnerable to dilution, the focus of a large literature since Fama and Miller (1972). Sovereign default differs from bankruptcy in that there is no liquidation of assets; to make the distinction with this mechanism even starker we abstract from partial repayment after a default. The potential for dilution in our environment comes solely via the role of maturity and prices in providing incentives to reduce debt. Partial liquidation also endows short-term bonds with implicit seniority. Brunnermeier and Oehmke (2013) show how this may induce a maturity rat-race that results in a collapse of the maturity structure. Interestingly, their mechanism is not operative when the liquidation value in bankruptcy is zero, which is the case of our environment.

The incentive to dilute existing long-term bond holders is also highlighted in recent quantitative models with long maturity debt (Hatchondo and Martinez, 2009, Chatterjee and Eyigungor, 2012, Arellano and Ramanarayanan, 2012). The Arellano and Ramanarayanan paper is particularly relevant, as it contains an active margin for maturity management at each period of time. Their analysis highlights that maturity structure plays two roles. The first is that in an environment of incomplete markets, maturity choice determines how the available assets span shocks, a feature which arises in incomplete-market models with perfect commitment (for example, Angeletos, 2002 and Buera and Nicolini, 2004). The second involves enforcement. In particular, maturity structure and the costs associated with default can be used to support a richer set of state-contingent repayments by increasing repayment incentives in certain paths. A main result of their quantitative analysis is that maturities shorten as the probability of default increases. A recent paper by Dovis (2012) sheds some light on this. Dovis generates a similar shortening of maturity through the spanning motive alone, relying on trigger strategies to handle enforcement. Another recent paper, Niepelt (2012), makes significant progress in casting maturity choice in an analytically tractable framework. It derives closed-form solutions that highlight the role of insurance via a covariance term. It also highlights that long-term bond prices are relatively elastic, a feature which plays an important role in our framework as well as the quantitative literature cited above. Another relevant paper is Broner et al. (2013), which was the first to focus attention on the shift to short-term debt during crisis in emerging markets. They proposed an explanation that is based on time varying risk premia, something that we rule out by construction by assuming risk neutral lenders.[2]

[2]See also the work of Perez (2013) for a more recent data analysis, covering a larger sample of emerging markets; as well for an alternative explanation based on asymmetric information.

Our analysis complements these papers by providing a transparent and tractable framework for analyzing maturity choice, by identifying the role of the maturity structure in the speed of deleveraging, and by explaining why an active use of long-term bonds shrinks the budget set of the sovereign. To do this, we consciously abstract from spanning by focusing on large shocks with a constant hazard rate of arrival, rather than the small fluctuations associated with business cycles. This simplification allows for a complete characterization of the equilibrium. Moreover, within our framework, we can easily introduce rollover crises as well as perform the analysis of Pareto efficient restructurings.

The sub-optimality of repurchasing long-term bonds on secondary markets is reminiscent of Bulow and Rogoff (1988, 1991). Their analysis turns on a finite amount of resources available to pay bond holders. In such a situation, a bond buyback concentrates the remaining bondholders' claim on this payout, and so drives up the price of bonds. Indeed, in this environment the sovereign would like to dilute existing bond holders by selling additional claims to this fixed recovery amount. The money raised would come in part at the expense of the previous bond holders, thus subsidizing the bond issue. The Bulow-Rogoff environment contains no incentive for the sovereign to pay down its debt, whether via a buy back or not reissuing short-term debt, and its three-period structure offers little scope for debt dynamics. In our environment, there is no liquidation value to debt which can be diluted; rather, the behavior of bond prices is due solely to the incentive effects of maturity choice.

Our discussion of rollover crises is related to the work of Calvo (1988), Cole and Kehoe (2000), and Aguiar et al. (2012). Our model allows the sovereign to actively manage the maturity structure of its debt, allowing us to analyze the tradeoff between saving (which is featured in Cole and Kehoe as well as Aguiar et al) and maturity management in the presence of rollover risk.

The remainder of the paper is organized as follows. Section 2 presents the general environment. Section 3 studies the benchmark model, which features a unique equilibrium, and states the main results. Section 4 shows how the results extend to an environment with coordination failures and the possibility of rollover crisis. Section 5 discusses the Pareto inefficiency of market equilibrium and how a restructuring can improve upon the market outcome. Section 6 discusses how the results generalize to a more general portfolio of maturities. A final section concludes.

2 Environment

We are interested in studying equilibria in which the economy faces uninsurable risk that may lead to endogenous default. In particular, our focus is on scenarios in which the risk of default is a first-order concern for both consumption-saving decisions as well as maturity choice. During tranquil periods sovereigns issue a range of maturities to smooth tax distortions; to provide a source of safe assets for savers; to facilitate payments systems; and to insure against fluctuations in tax revenues, output or interest rates. However, in the midst of a sovereign debt crisis these considerations are to a large extent dominated by a sovereign's need to issue new debt to skeptical investors, to roll over outstanding debt, and to reduce the outstanding stock of debt in a credible (that is, time consistent) manner. We therefore build a model that transparently isolates the role of maturity choice in determining whether and how a sovereign deleverages under the threat of default.

Consider a small open economy in a discrete time, infinite horizon environment, with time indexed by $t \in \{0, 1, ..., \}$. There is a single, freely tradable, numeraire consumption good, of which the economy receives a constant endowment of y each period.[3]

The sovereign makes economic decisions on behalf of the small open economy. The sovereign's preferences over consumption streams are characterized by the following utility function:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where $\beta \in (0, 1)$ and u is bounded, strictly increasing, strictly concave, and satisfies the Inada conditions.

The sovereign trades financial claims with the rest of the world, which is populated by competitive, risk-neutral agents who share the sovereign's discount factor β. We assume that foreign agents ("creditors") are willing to borrow and lend at an expected interest rate $R = 1 + r = \beta^{-1}$. Given the small open economy assumption, we assume that the rest of the world's resource constraint is irrelevant as long as the creditors break even in expectation.

Available international assets consist of two types of non-contingent bonds. There exists a short-term bond which calls for the sovereign to pay R units of a numeraire tradable good in the next period. There is also a long-term bond, in this case a perpetuity. In Section 6 we show that the results extend directly to the case with a more general maturity structure, which will be important when considering empirical implications for the average maturity of outstanding debt versus that of new issuances. This bond calls for the sovereign to pay r

[3] Note that the endowment stream is not subject to fluctuations. It is well known that in models with incomplete markets and shocks to output or government expenditure, maturity choice can be used to partially or fully replicate a full set of state contingent assets (Angeletos, 2002, Buera and Nicolini, 2004). As noted above, our focus is on the role of maturity in providing incentives to save and in exposing the economy to rollover risk. To maintain a clear distinction between incentives versus spanning, we abstract from income fluctuations.

every period, and never matures. The choice of a coupon equal to r is a normalization that implies that the equilibrium risk-free price of the long-term bond is 1. We assume that debt positions are non-negative.

There is limited enforcement of claims on the small open economy. The economy enters period t with outstanding short- and long-term debt positions $b_{S,t}$ and $b_{L,t}$, respectively. To comply with the terms of the debt contract, the sovereign is obligated to pay in aggregate $(1+r)b_{S,t} + rb_{L,t}$. If the sovereign opts not to make this payment, the country is in default. A fundamental issue in sovereign debt markets concerns the limited ability of creditors to enforce contracts with a sovereign government. We assume that in case of the default, the payoff to the sovereign is captured by the value V^D, which we restrict to be such that $u(0)/(1-\beta) < V^D < u(y)/(1-\beta)$.[4] An important assumption is that V^D does not depend on the quantity of debt before default.[5]

We capture an important feature of real-world sovereign bond markets by allowing the consequences of default to vary stochastically over time. Shifts in political sentiment regarding default as well as the willingness of foreign courts to enforce bond contracts imply that movements in V^D can be a source of risk for creditors. Recent examples of shifts in enforcement include the pressure put on euro-zone banks by regulators to write down claims against Greece in 2012 as well as a string of US court decisions regarding Argentina's restructured debt and hold-out creditors.

We capture this risk by assuming that V^D is a random variable with two possible outcomes, $V^D \in \{\underline{V}^D, \overline{V}^D\}$, with $\overline{V}^D > \underline{V}^D$. The realization $\overline{V}^D$ is one in which default is relatively attractive, which we shall refer to as the *weak enforcement* regime, while $\underline{V}^D$ represents a *strong enforcement* regime. We assume that the initial state is $\underline{V}^D$, and the regime switches to $\overline{V}^D$ with a constant probability λ each period. Once $\overline{V}^D$ is realized, $V^D = \overline{V}^D$ with probability one thereafter.[6]

By slightly altering the timing of bond-issuances and default in this environment, we can study equilibria with and without self-fulfilling debt crises. In the next section, we characterize this environment with the Eaton and Gersovitz (1981) timing, a situation that leads to a unique equilibrium (without rollover crises). With this benchmark in hand, in

[4]We could be more explicit and assume that the default is "punished" by financial autarky as well as an output cost. There may also be (positive or negative) political consequences for the incumbent government that chooses to default. In this way, we could write an explicit function for the value V^D. However, the default decision of the sovereign will be determined by comparing the value from repayment versus the resulting value V^D, and therefore the latter value is a sufficient statistic for the sovereign's incentives prior to default.

[5]While partial default and renegotiation are important issues in sovereign defaults, for tractability reasons we leave these aside.

[6]The fact that $\overline{V}^D$ is an absorbing state is not necessary for the results, but simplifies the exposition.

Section 4 we consider self-fulfilling crisis equilibria by switching to the timing of Cole and Kehoe (2000). We show that our results are robust to this extension, as well as generating a stronger intuition for the role of the maturity structure in avoiding self-fulfilling runs.

3 The Benchmark Model

We consider Markov equilibria in which prices and policies are functions of outstanding debt and the default value: (b_S, b_L, V^D), as well as whether the sovereign has defaulted in the past. We shall use bold-face $\boldsymbol{b} \equiv (b_S, b_L)$ as short-hand for the outstanding portfolio of debt. The timing of a period is depicted in Figure 1 and proceeds as follows. Unless otherwise noted, we assume the sovereign has not defaulted in a previous period and omit the credit-history state from the notation.

The sovereign enters the period with outstanding short-term and long-term debt $\boldsymbol{b} = (b_S, b_L) \in \mathcal{B} \equiv [0, \infty) \times [0, \infty)$. At the start of the period, V^D is realized and endowment y is received. After observing V^D, the sovereign decides whether it will default in the current period or not. If it defaults, it receives V^D. If it does not default, it auctions b'_S short-term and $b'_L - b_L$ long-term bonds, pays interest and principal as necessary on outstanding bonds, and consumes. This timing is that of Eaton and Gersovitz (1981) and a large subsequent literature, but it does embed an important assumption. In particular, at the time of issuing new bonds the sovereign has committed not to default within the current period. This timing contrasts with that of Cole and Kehoe (2000), in which new bonds are issued before the current period's default decision is made. We shall take up a version of the latter timing in Section 4 when we consider rollover crises. As we shall see, the Eaton-Gersovitz timing rules out some equilibria that can be supported under the alternative timing.

3.1 Equilibrium Definition

Recall that the sovereign enters a period with $\boldsymbol{b}$ and observes V^D. If it does not default, it can issue bonds at equilibrium bond prices $q_j(\boldsymbol{b}, V^D, \boldsymbol{b}')$, $j = S, L$, where $\boldsymbol{b}'$ is the amount of debt outstanding at the end of the period.[7] Denote the value function of a sovereign which has not previously defaulted and does *not* default in the *current* period by $V(\boldsymbol{b}, V^D)$. In particular,

[7]Note that the sovereign is "large" in regard to its own sovereign debt and internalizes the fact that equilibrium bond prices depend on the amount of bonds outstanding.

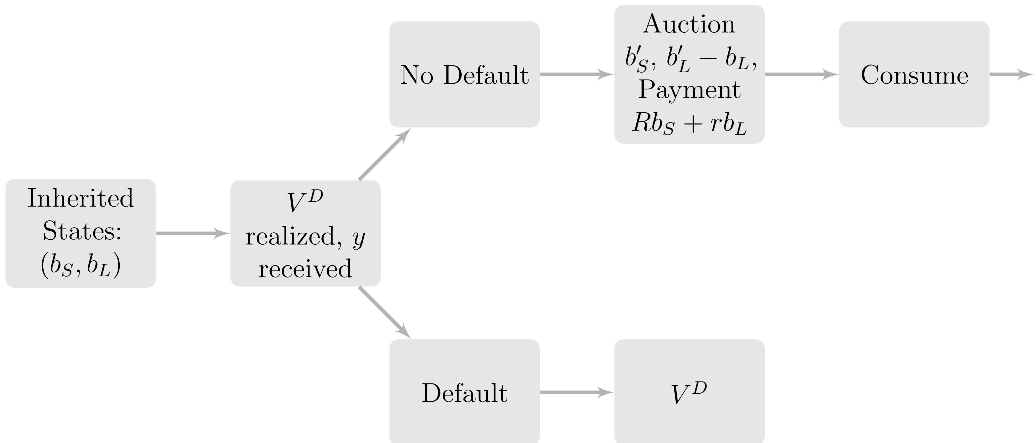

Figure 1: Timing within a Period in the Benchmark Model.

for $\boldsymbol{b} \in \mathcal{B}$ and $V^D \in \left\{\underline{V}^D, \overline{V}^D\right\}$, this value function satisfies the Bellman equation:

$$V(\boldsymbol{b}, V^D) = \max_{\{c \geq 0, \boldsymbol{b} \in \mathcal{B}\}} \left\{u(c) + \beta \mathbb{E}\left[\max\langle V(\boldsymbol{b}', V^{D\prime}), V^{D\prime}\rangle \middle| V^D\right]\right\} \tag{P1}$$

$$\text{subject to:}$$

$$c + (1+r)b_S + rb_L \leq y + q_S(\boldsymbol{b}, V^D, \boldsymbol{b}')b'_S + q_L(\boldsymbol{b}, V^D, \boldsymbol{b}')(b'_L - b_L).$$

where $\boldsymbol{b}' = (b'_S, b'_L)$. The expectation operator on the continuation value is over next period realizations of $V^{D\prime}$, conditional on the current period's V^D, and the continuation value is the maximum over the no-default and default values next period. Let $C(\boldsymbol{b}, V^D)$, $B_S(\boldsymbol{b}, V^D)$ and $B_L(\boldsymbol{b}, V^D)$ denote the optimal policies for c, b'_S and b'_L, respectively. The default decision depends on whether the above value is greater or less than V^D. Let $D(\boldsymbol{b}, V^D)$ be the default policy function. That is, $D(\boldsymbol{b}, V^D) = 1$ if $V^D > V(\boldsymbol{b}, V^D)$, and zero otherwise, where we impose the tie-breaking assumption that the sovereign repays when indifferent. We will also assume that the sovereign does not issue bonds at zero price, as that is never strictly optimal.[8] The equilibrium value function at the beginning of the period is $\max\langle V(\boldsymbol{b}, V^D), V^D\rangle$.

To characterize the sovereign's problem fully, we need to know more about the bond price schedules. In particular, competition among the foreign creditors guarantee that that

[8] That is, we rule out $\boldsymbol{b}'$ such that $q_S(\boldsymbol{b}, V^D, \boldsymbol{b}') = 0$. This assumption resolves an indeterminacy that arises when the government is indifferent between defaulting today and tomorrow.

creditors problem can be characterized by the break even (BE) conditions:

$$\begin{aligned} q_S(\boldsymbol{b}, V^D, \boldsymbol{b}') &= \mathbb{E}\left[1 - D(\boldsymbol{b}', V^{D\prime}) \middle| V^D\right] \\ q_L(\boldsymbol{b}, V^D, \boldsymbol{b}') &= \mathbb{E}\left[\left(1 - D(\boldsymbol{b}', V^{D\prime})\right)\left(\frac{r + q_L'}{1+r}\right) \middle| V^D\right], \end{aligned} \tag{BE}$$

where q_L' in the long-term bond price is shorthand for the long-term bond price next period conditional on no default, $\boldsymbol{b}'$, and the equilibrium policy functions of the sovereign, $B_j(\boldsymbol{b}', V^{D\prime}), j = S, L$, that determine debt positions at the end of next period. Note that the timing from Figure 1 implies that bond purchasers are not vulnerable to default risk in the period they purchase bonds and therefore the initial state $\boldsymbol{b}$ is not relevant for bond prices, conditional on $\boldsymbol{b}'$ and V^D. We therefore can drop that state from the bond price notation and simply write $q_j(V^D, \boldsymbol{b}')$, $j = S, L$, where $\boldsymbol{b}'$ is the end-of-period bond portfolio. Interior positions on the part of creditors require that the break even conditions hold with equality. We rule out the possibility of bubbles in the perpetuities by considering equilibria with $q_L \leq 1$, where 1 is the price of a risk-free bond with coupon r. We also rule out Ponzi schemes.

We now proceed to define equilibrium in this environment:

Definition 1. *A Markov Perfect Equilibrium consists of policy functions C, B_S, B_L, and D, and pricing schedules q_S and q_L, such that for all debt positions $\boldsymbol{b} \in \mathcal{B}$ and $V^D \in \{\underline{V}^D, \overline{V}^D\}$, and absent a prior default: (i) the policy functions C, B_S, and B_L, solve the sovereign's problem (P1) conditional on q_S and q_L and the No-Ponzi condition; (ii) D is an indicator function that takes one if $V(\boldsymbol{b}, V^D) < V^D$ and zero otherwise; and (iii) the creditors' break-even conditions (BE) are satisfied with $q_i \in [0, 1], i = S, L$, given the sovereign's policy functions.*

Although, there are multiple Markov equilibrium in this environment, we will show below that, in effect, they are all identical: that is, bond prices and consumption are uniquely pinned down up to a tie-breaking assumption required by the discreteness of time. As we discuss below, the multiple equilibria emerges because of the possibility that the portfolio allocation may be (locally) indeterminate, an expected outcome given that the bonds' payouts, in principle, span each other.

3.2 Characterizing Equilibria

In characterizing equilibria, it is useful to divide the state space for debt into three regions. For low levels of debt, the sovereign will not default regardless of the realization of V^D. We shall refer to that region as the "no-default" zone, and denote it by $\boldsymbol{ND}$:

$$\boldsymbol{ND} = \{\boldsymbol{b} \in \mathcal{B} \big| V(\boldsymbol{b}, V^D) \geq V^D, \forall V^D\}.$$

Note that this region depends on the value function V and so is not independent of equilibrium prices and policies. As we proceed, we shall explicitly define the equilibrium contents of $\boldsymbol{ND}$.

At intermediate levels of debt, the sovereign will default if the outside option is high (that is, $V^D = \overline{V}^D$), but not if it is low. We shall refer to this region as the "crisis" region, given that there is a positive probability of default, and denote it by $\boldsymbol{C}$:

$$\boldsymbol{C} = \left\{\boldsymbol{b} \in \mathcal{B} \middle| V(\boldsymbol{b}, \underline{V}^D) \geq \underline{V}^D \quad \& \quad V(\boldsymbol{b}, \overline{V}^D) < \overline{V}^D \right\}.$$

Finally, at high enough levels of debt, the sovereign finds it optimal to default even if the default payout is low. If the initial debt is in this "default" zone, the sovereign immediately defaults. Denoting this region by $\boldsymbol{D}$, we have:

$$\boldsymbol{D} = \left\{\boldsymbol{b} \in \mathcal{B} \middle| V(\boldsymbol{b}, \underline{V}^D) < \underline{V}^D \quad \& \quad V(\boldsymbol{b}, \overline{V}^D) < \overline{V}^D \right\}.$$

This region is of little interest as in equilibrium there is no feasible path that allows the sovereign to accumulate so much debt. The interesting demarcation is between the no-default and crisis zones, and how debt and maturity structure evolves in each region.

Before proceeding to flesh out these regions, we first show that in any Markov equilibria, the state space is partitioned into these three regions:

Proposition 1. *In any Markov Perfect Equilibrium, the sets $\boldsymbol{ND}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are non-empty, disjoint, and $\boldsymbol{ND} \cup \boldsymbol{C} \cup \boldsymbol{D} = \mathcal{B}$.*

Moreover, these regions have a natural ordering such that debt is increasing as we move from $\boldsymbol{ND}$ to $\boldsymbol{C}$ to $\boldsymbol{D}$. In particular, if $(b_S^0, b_L^0) \in \boldsymbol{ND}$, then there exists $b_S^2 > b_S^1 > b_S^0$ such that $(b_S^1, b_L^0) \in \boldsymbol{C}$ and $(b_S^2, b_L^0) \in \boldsymbol{D}$. Similarly, there exists $b_L^2 > b_L^1 > b_L^0$ such that $(b_S^0, b_L^1) \in \boldsymbol{C}$ and $(b_S^0, b_L^2) \in \boldsymbol{D}$. This follows directly from the monotonicity and continuity of the value function, which is proved in the appendix (Lemma A.1).

3.2.1 The No-Default Region

The no-default region is straightforward to characterize. If the sovereign begins the period with $\boldsymbol{b} \in \boldsymbol{ND}$ it has no incentive to exit. In particular, to exit the region it must fully compensate new bondholders for the increased risk of default. The capital loss suffered by the existing long-term bond holders is of no benefit to the sovereign, a point we will discuss in detail below. Therefore, for any realization of V^D the sovereign will not default in a period in which $\boldsymbol{b} \in \boldsymbol{ND}$. This implies that $q_S(V^D, \boldsymbol{b}') = 1$ for all $\boldsymbol{b}' \in \boldsymbol{ND}$. It can therefore simply stay put by rolling over its short-term debt at risk-free prices and paying r on its perpetuities. In fact, it cannot do better:

Proposition 2. ***No-Default Region**: Define $\overline{B}$ by $u(y - r\overline{B}) = (1-\beta)\overline{V}^D$. In any Markov Perfect Equilibrium the no-default region is defined as:*

$$\boldsymbol{ND} = \left\{ \boldsymbol{b} \in \mathcal{B} \,\middle|\, b_S + b_L \leq \overline{B} \right\}.$$

Moreover, for all $\boldsymbol{b} = (b_S, b_L) \in \boldsymbol{ND}$ and $V^D \in \left\{ \underline{V}^D, \overline{V}^D \right\}$, the equilibrium value function is

$$V(\boldsymbol{b}, V^D) = \frac{u(y - r(b_S + b_L))}{1 - \beta};$$

equilibrium prices satisfy

$$q_S(V^D, \boldsymbol{b}) = q_L(V^D, \boldsymbol{b}) = 1;$$

and equilibrium policy functions satisfy

$$B_S(\boldsymbol{b}, V^D) + B_L(\boldsymbol{b}, V^D) = b_S + b_L.$$

This proposition contains a number of statements about equilibrium behavior in the no-default region. In terms of pricing, the short-term bond price is one (the risk free price) by definition: the no-default zone is defined as a region in which the sovereign will not default the next period regardless of V^D. The fact that long-term bonds are also risk free reflects that the no-default zone is an absorbing region. Once there the sovereign will never exit, and so long-term bonds issued in the no-default zone are never exposed to default risk over the infinite life of the perpetuity. As both bonds are risk free, they are perfect substitutes, which is reflected in the fact that the value function depends on the sum of short-term and long-term debt. Moreover, this value is simply what the sovereign obtains by servicing interest payments and keeping total debt stationary, which is reflected in the associated policy functions. The optimality of this policy rests on the fact that $\beta = R^{-1}$. The value function also pins down the boundary of the no-default zone. Faced with risk-free pricing, the optimal policy is to keep debt constant. The generated no-default value is $u(y - r(b_S + b_L))/(1 - \beta)$. Now note that if this value is strictly less than $\overline{V}^D$, the sovereign will default if $V^D = \overline{V}^D$ and so this cannot be part of the no-default region. Similarly, if $u(y - r(b_S + b_L))/(1-\beta) > \overline{V}^D$, a continuity argument implies this cannot be the boundary of the no-default region.

Note that long-term bond holders experience a capital loss if the sovereign exits $\boldsymbol{ND}$, but this does not provide an incentive for the sovereign to borrow. The capital loss suffered if the government were to exit the no-default region does not benefit the sovereign. This highlights that the risk of default is not a zero-sum game. For example, suppose that $b_S + b_L = \bar{B}$, and thus the sovereign is indifferent to defaulting or not. Consider a one-time deviation from the government's equilibrium strategy in which it flips a coin to decide whether to default or not. This imposes a cost on bond holders, but provides no benefit to the government given the indifference between repaying and defaulting at $\bar{B}$. This underlies why exiting $\boldsymbol{ND}$ at the expense of existing bond holders is never optimal in equilibrium.

3.2.2 The Crisis Region

Having characterized the no-default region, we now turn to the crisis zone. Recall that the crisis zone is defined as the region in which the sovereign defaults if the weak enforcement regime, $V^D = \overline{V}^D$, is realized, but not otherwise. Characterizing the equilibrium in the crisis region when $V^D = \overline{V}^D$ is therefore straightforward, as by definition the country defaults if this state is realized. In particular, for $\boldsymbol{b} \in \mathcal{B} - \boldsymbol{ND}$ (that is, the complement of $\boldsymbol{ND}$ in $\mathcal{B}$),

$$q_S(\overline{V}^D, \boldsymbol{b}) = q_L(\overline{V}^D, \boldsymbol{b}) = 0.$$

The more complex case is when $V^D = \underline{V}^D$.In what follows, we focus on this case and suppress the notation that the initial state is $V^D = \underline{V}^D$ when possible.

The equilibrium price for short term bonds is straightforward to characterize:

$$q_S(\boldsymbol{b}) = \begin{cases} 1 & \text{if } \boldsymbol{b} \in \boldsymbol{ND} \\ 1-\lambda & \text{if } \boldsymbol{b} \in \boldsymbol{C} \\ 0 & \text{if } \boldsymbol{b} \in \boldsymbol{D} \end{cases} \tag{2}$$

where as mentioned above we suppress $V^D = \underline{V}^D$ and the argument $\boldsymbol{b}$ is the end-of-period bond position. If end-of-period bonds are in the crisis region, the sovereign defaults next period with probability $1-\lambda$. The other two regions pose no uncertainty, and so the prices are 1 in $\boldsymbol{ND}$ and 0 in $\boldsymbol{D}$.

To characterize long-term bond prices in $\boldsymbol{C}$, we will ignore the strategies whereby the sovereign reaches the set $\boldsymbol{D}$, as these can never be optimal, and thus restrict attention to strategies that keep the debt portfolio in the set $\boldsymbol{C}$ or exit to $\boldsymbol{ND}$. Let us consider then the break-even condition for creditors given the sovereign's equilibrium policy functions. Starting from an initial $\boldsymbol{b}$, we can iterate on the debt-issuance policies B_i, $i = S, L$, assuming $\overline{V}^D$ has not been realized, to determine the number of periods until the sovereign's portfolio reaches the no-default zone (if ever). Denote this time until exit from $\boldsymbol{C}$ by $T(\boldsymbol{b})$. That is, if $\boldsymbol{b} \in \boldsymbol{ND}$, then $T(\boldsymbol{b}) = 0$; if $\boldsymbol{b} \in \boldsymbol{C}$, and $(B_S(\boldsymbol{b}, \underline{V}^D), B_L(\boldsymbol{b}, \underline{V}^D)) \in \boldsymbol{ND}$, then $T = 1$; and so on. Let $B_i^\tau(\boldsymbol{b}), i = S, L$, denote the bond positions reached by iterating on the policy functions starting from $\boldsymbol{b}$ for τ times, along a path such that $V^D = \underline{V}^D$ at each step. Then,

$$T(\boldsymbol{b}) = \min \left\{ \tau \in \{0, 1, ...\} \,|(B_S^\tau(\boldsymbol{b}), B_L^\tau(\boldsymbol{b})) \in \boldsymbol{ND} \right\}.$$

If no such minimum exists, then $T(\boldsymbol{b}) = \infty$.

To obtain prices, we solve the creditors' break even constraints forward starting from $\boldsymbol{b} \in \boldsymbol{C}$ (that is, $T \geq 1$) and, by Proposition 2, we can impose the boundary condition that $q_L = 1$ at the start of the T'th period:

$$\begin{aligned} q_L(\boldsymbol{b}) &= \left(\frac{1-\lambda}{1+r}\right)(r + q_L(B_S(\boldsymbol{b}), B_L(\boldsymbol{b}))) \\ &= r \sum_{t=1}^{T(\boldsymbol{b})} \left(\frac{1-\lambda}{1+r}\right)^t + \left(\frac{1-\lambda}{1+r}\right)^{T(\boldsymbol{b})}. \end{aligned} \tag{3}$$

Although the first line recursion assumes $T(\boldsymbol{b}) \geq 1$ (hence, the $1 - \lambda$ in the discount factor), nevertheless the final line correctly implies $q_L = 1$ for $T = 0$.

Note that all that is relevant for bond prices is the speed with which the sovereign eliminates the possibility of default, not the particular path (or maturity structures) chosen. When auctioning off long-term debt, the sovereign would like to pledge a quick exit from the crisis zone, so as to raise the value of the issuances. When buying debt back, the sovereign would like to pledge the opposite, as to reduce the value of the outstanding bonds. However, such a pledge must be credible in an environment of limited commitment. We will explore below how the requirement of time consistency brings maturity choice back into the picture.

Why Save? The "Price Effect"

The first issue to establish is whether and why the sovereign would save at all from the Crisis region to the No-Default region. In particular, suppose the sovereign is currently under the strong enforcement regime ($V^D = \underline{V}^D$) and $\boldsymbol{b} \in \boldsymbol{C}$. Would it be better off waiting for the weak regime and then defaulting, rather than saving its way out of the region? To see why this may be a tempting option, recall that by definition, $V(\boldsymbol{b}, V^D) < \bar{V}^D$ for $\boldsymbol{b} \in \boldsymbol{C}$, and so the sovereign's value is higher if $\bar{V}^D$ is realized and it chooses to default. Is there any incentive to save to remove this possibility?

To explore this question, consider the value of a sovereign that remains at $\boldsymbol{b} \in \boldsymbol{C}$. The price of short-term bonds in the crisis zone is $q_S = 1 - \lambda$, and letting an infinity denote the policy of remaining in $\boldsymbol{C}$ forever, the budget constraint implies consumption c^∞ is:

$$\begin{aligned} c^\infty &= y - (1+r)b_S - rb_L + (1-\lambda)b_S \\ &= y - r(b_S + b_L) - \lambda b_S. \end{aligned}$$

To see whether the sovereign can improve on this policy, consider a policy in which the sovereign pays down debt to reach the no-default zone next period. Let $\boldsymbol{b}' = (b'_S, b'_L) \in \boldsymbol{ND}$ denote end-of-period debt such that $b'_S + b'_L = \bar{B}$. Define $\Delta \equiv b_S + b_L - \bar{B}$ as the amount of debt that needs to be repaid this period to reach $\boldsymbol{ND}$. Recall that $q_i(\boldsymbol{b}') = 1, i = S, L$, for $\boldsymbol{b}' \in \boldsymbol{ND}$. The consumption in the initial period for this one-step policy, c^1, is:

$$\begin{aligned} c^1 &= y - (1+r)b_S - rb_L + b'_S + b'_L - b_L \\ &= y - r(b_S + b_L) - \Delta. \end{aligned}$$

Taking differences, we have:

$$c^1 - c^\infty = -\Delta + \lambda b_S.$$

As $\Delta \to 0$, we have $c^1 - c^\infty > 0$ as long as $b_S > 0$. That is, if the sovereign has short-term debt outstanding, it is clearly better off paying down its debt. Initial consumption is greater ($c^1 \to y - r\bar{B} > c^\infty$) and the continuation value is greater.[9] Therefore, saving is optimal in the neighborhood just outside the no-crisis zone as long as short-term debt is not zero.[10]

The fact that the sovereign saves in an environment in which default occurs with an increase in welfare reflects a "price effect." In particular, in the crisis zone the sovereign must compensate bond holders for the risk of default. The subtlety is that bond prices are actuarially fair, so why does this matter? It is not risk aversion, as our analysis simply compared the level of consumption under two alternative policies, and did not require concavity. The answer is that the bond holders must be compensated for the loss of all (newly

[9]To see that the continuation value is greater, note that $V(\boldsymbol{b}, V^D) = u(y - r(b_S + b_L))/(1-\beta) = \overline{V}^D$ is the continuation value for c^1. The continuation value for c^∞ is $\sum_{t=1}^\infty \beta^t (1-\lambda)^t u(c^\infty) + \lambda \sum_{t=1}^\infty \beta^t (1-\lambda)^{t-1} \overline{V}^D$ $= \frac{u(c^\infty)}{1-\beta(1-\lambda)} + \frac{\beta\lambda}{1-\beta(1-\lambda)} V(\boldsymbol{b}, V^D) < V(\boldsymbol{b}, V^D)$, where the last inequality follows if $c^\infty < y - r(b_S + b_L)$.

[10]We have not established that exiting in one step is necessarily optimal, only that it dominates never exiting in a neighborhood of the no-default zone. In what follows, we shall see that as we move further away from the boundary of the no-default zone, the incentive to exit remains but the optimal number of steps to exit increases.

issued) debt claims: λb_S. However, the gain for the sovereign in the event of default is not this amount, but rather $\overline{V}^D - V(\boldsymbol{b}, \underline{V}^D)$, which is arbitrarily small at the boundary of the safe zone. Thus, the realization of $\overline{V}^D$ and default is a large loss for bond holders, but a small utility gain for the sovereign.

In the crisis zone, the sovereign is being forced to compensate short-term bond holders for their potential loss, with no offsetting benefit to itself. The bondholders' risk is priced into short-term bonds on a continuous basis, while it is built into long-term bond payments at issuance. In this sense, short-term debt presents a variable cost for remaining in the crisis zone, while long-term bonds represent a sunk cost. If $b_S = 0$, the only liabilities are perpetuities and the steady state policy requires no new bond issuance. In this case, the sovereign has no incentive to save out of the crisis zone as it never issues new debt. The fact that short-term bonds provide a greater incentive to save than long-term bonds will reappear throughout the analysis that follows.

Managing Maturity while Deleveraging

The preceding analysis established that the presence of short-term debt provides an incentive to exit the crisis region. We now turn to the question of managing maturity structure during the deleveraging process. The following is a major result of the analysis:

Theorem 1. *In any Markov Perfect Equilibrium, never issuing or repurchasing long term debt is part of a weakly optimal strategy. That is, a policy with $b'_L = b_L$ achieves the maximum of Problem* (P1).

We use the proof of the theorem to highlight the forces at work in debt deleveraging in our environment.[11] It is useful to consider an alternative maximization problem whereby the sovereign commits to an exit time, but is restricted to trading only short-term bonds along the transition path. We first show that the value from this program weakly dominates the equilibrium value when the exit time is the equilibrium exit time. That is, conditional on exit time, remaining passive in long-term bond markets is without loss. This is Lemma 1.

Of course, commitment to an exit time is a strong assumption. The second lemma shows that commitment to an exit time is superfluous when the sovereign does not issue or

[11] An alternative proof of this result can be obtained by extending the decentralization result in Hopenhayn and Werning (2008) by adding legacy debt (and a legacy principal) and considering the optimal contract with a new principal. We thank Ivan Werning for noting this connection. We pursue an alternative line of reasoning using a replication argument for expositional reasons. Our approach highlights how the initial portfolio of maturities affects the government's equilibrium budget set for *any* equilibrium bond price schedules that satisfy the break-even constraint. The argument also provides a bridge to the discussion of why trading long-term bonds is sub-optimal and is easily adapted to the case with coordination failures in a subsequent section.

repurchase long-term bonds. This is because only long-term bonds are sensitive to promises of future actions, while short-term bond prices are pinned down by debt positions chosen within the current period. If the sovereign does not trade long-term bonds, then the commitment solution remains time consistent in the absence of commitment. This is Lemma 2.

Before stating the lemmas, we introduce the sovereign's problem conditional on exit time $T \geq 1$, imposing the constraint that $b_{L,t}$ is constant. In particular, let $\boldsymbol{b} = (b_S, b_L)$, and let

$$W(\boldsymbol{b}, T) = \max_{\left\{b_{S,T} \in \mathbb{R}, \{c_t\}_{t=0}^{T-1}\right\}} \left\{ \sum_{t=0}^{T-1} \beta^t (1-\lambda)^t u(c_t) + \beta^T (1-\lambda)^{T-1} \left(\frac{u(y - r(b_{S,T} + b_L))}{1-\beta} \right) + \sum_{t=1}^{T-1} \beta^t (1-\lambda)^{t-1} \lambda \overline{V}^D \right\}, \quad \text{(PW)}$$

subject to:

$$\begin{aligned} b_S &\leq (1+r)^{-1} \left(\sum_{t=0}^{T-1} \left(\frac{1-\lambda}{1+r} \right)^t (y - c_t - r b_L) + \left(\frac{1-\lambda}{1+r} \right)^{T-1} b_{S,T} \right), \\ b_{S,t} &> \overline{B} - b_L \text{ for all t} < T, \\ b_{S,T} &\leq \overline{B} - b_L. \end{aligned}$$

and $c_t \geq 0$ for all t where $b_{S,t}$ is defined recursively as $b_{S,t+1} = (y - c_t - r b_L - R b_{S,t})/(1-\lambda)$ with $b_{S,0} = b_S$.[12] Note that we allow short-term debt positions be negative in the W problem. We also allow for $T = \infty$ in the W problem by taking the limit as $T \to \infty$, and replacing the last constraint with a no-Ponzi condition.

The objective function through period T is akin to a "perpetual youth" problem in which the sovereign "dies" with constant hazard λ and receives $\bar{V}^D$, and otherwise receives the no-default-region value if it survives to period T. The budget constraint is the discounted sum of the sequential constraints $c_t = y - (1+r)b_{S,t} - r b_L + (1-\lambda) b_{S,t+1}$ for $t < T-1$, and $c_{T-1} = y - (1+r) b_{S,T-1} - r b_L + b_{S,T}$ for $t = T-1$. If T is not feasible starting from (b_S, b_L), that is, requires negative consumption, then it suffices to set $W(b_S, b_L, T) = -\infty$.

The next lemma states that the solution to the above problem evaluated at the equilibrium exit time weakly dominates the equilibrium value function.

Lemma 1. *Suppose $T(\boldsymbol{b})$ is an equilibrium time-until-exit. Then $V(\boldsymbol{b}, \underline{V}^D) \leq W(\boldsymbol{b}, T(\boldsymbol{b}))$ for any $\boldsymbol{b} \in \boldsymbol{C}$.*

[12] The constraint that $b_{S,t} > \overline{B} - b_L$ would not bind in a solution to the W problem.

The proof of this lemma (see Appendix) uses the fact that the sovereign's budget set in both the W and V problems is determined by the initial discounted expected payments to bond holders. Holding constant T and the initial debt position is the same as holding constant expected discounted payments, and the precise path chosen is irrelevant for the budget set conditional on T. Therefore, the sequence which keeps $b_{L,t}$ constant does just as well as the candidate equilibrium sequence.

However, the premise of the W value function is that the sovereign can commit to T. This begs the question of whether the sequence of bond positions is time consistent absent commitment. Before showing this, let us argue that the optimal solution to the W problem does not feature non-negative debt positions:

Claim 1. *For any $(b_S, b_L) \in \boldsymbol{C}$ such that $b_L > \overline{B}$, it follows that $\{\infty\} = \arg\max_T W(b_S, b_L, T)$.*

This result implies that, starting from any $(b_S, b_L) \in \boldsymbol{C}$, in the solution to the problem $\max_T W(b_S, b_L, T)$, if the sovereign chooses to save, then it must be that $b_L < \overline{B}$ and thus, the implied short-term debt positions from the W problem are reduced every period but will always remain positive (that is, $b_{S,t} \geq b_{S,T} = \overline{B} - b_L > 0$).

The next lemma uses the fact that allocation that solves the best W problem is feasible in any equilibrium to state a converse of Lemma 1:

Lemma 2. *In any equilibrium, $V(\boldsymbol{b}, \underline{V}^D) \geq \sup_{T\geq 1} W(\boldsymbol{b}, T)$ for any $\boldsymbol{b} \in \boldsymbol{C}$.*

There are two key elements to the proof of this lemma. The first is that an allocation that solves problem W for a given T has the feature that the issuances of short term bonds are strictly decreasing over time and hence the implied debt positions remains in region $\boldsymbol{C}$ until exit. The second is that we know the equilibrium prices of the short term debt (they are equal to $1 - \lambda$ as long as we remain in $\boldsymbol{C}$, and 1 when we exit). It follows then, that we can compute the cost *at equilibrium prices* of an allocation that solves problem W and establish that such an allocation is feasible for the equilibrium problem. Hence, the equilibrium value function cannot be lower than $W(\boldsymbol{b}, T)$. Note that we can ignore the prices of the long-term bond in this argument, as the country does not issue them nor does it buy them back.

It is interesting to highlight how the above simple arguments fail if we were to restrict attention to strategies that use only long-term bonds. In this case, a version of Lemma 1 will still hold: if the sovereign commits to an exit time it is irrelevant for its welfare whether it uses short-term or long-term bonds to achieve this goal. However, we do not know the *equilibrium* prices of the long-term bonds, and so cannot establish that the resulting W-problem allocation is feasible in equilibrium. That is, Lemma 2 fails. And for good reason, as we discuss in Section 3.5, the optimal consumption allocation in general cannot be afforded in equilibrium with strategies that rely on trading long-term bonds.

Combining Lemmas 1 and 2, we have that in any equilibrium, $V(\boldsymbol{b}, \underline{V}^D)$ can be achieved by setting $b_{L,t+1} = b_{L,t}$ until exit from the crisis zone, which is the result of Theorem 1.

3.3 The Equilibrium Value Function

Theorem 1 allows us to pin down the crisis zone value function for any equilibrium. In particular, $V(\boldsymbol{b}, \underline{V}^D) = \sup_T W(\boldsymbol{b}, T)$ for $\boldsymbol{b} \in \boldsymbol{C}$, and we can therefore characterize the equilibrium value function by analyzing the W problem (PW).

The first thing to note is that at the time of exit, $b_{S,T} = \overline{B} - b_L$. That is, the sovereign does not "over save" in exiting the crisis zone.[13] Conditional on T, problem (PW) is a simple consumption-savings problem with an effective discount factor of $\beta(1-\lambda) = \frac{1-\lambda}{1+r}$, which equals the effective interest rate on short-term debt. Therefore, consumption will be constant while in the crisis zone. In particular, define $C^T(\boldsymbol{b})$ as consumption conditional on exit in $T \geq 1$ periods starting from $\boldsymbol{b} \in \boldsymbol{C}$. Holding consumption constant and evaluating the summation in the budget constraint of problem (PW) we have:

$$C^T(\boldsymbol{b}) = y - rb_L + \left(\frac{1-\beta(1-\lambda)}{1-\beta^T(1-\lambda)^T}\right)\left(\beta^{T-1}(1-\lambda)^{T-1}(\overline{B} - b_L) - (1+r)b_S\right). \tag{4}$$

Substituting into the objective function in (PW), we have:

$$\begin{aligned} W(\boldsymbol{b}, T) = &\left(\frac{1-\beta^T(1-\lambda)^T}{1-\beta(1-\lambda)}\right) u\left(C^T(\boldsymbol{b})\right) + \beta^T(1-\lambda)^{T-1}\left(\frac{u(y - r\overline{B})}{1-\beta}\right) \\ &+ \lambda\beta\left(\frac{1-\beta^{T-1}(1-\lambda)^{T-1}}{1-\beta(1-\lambda)}\right)\overline{V}^D. \end{aligned} \tag{5}$$

The value function $V(\boldsymbol{b}, \underline{V}^D) = \sup_T W(\boldsymbol{b}, T)$, which is a maximization over one argument. The usefulness of Theorem 1 here is that determining the equilibrium value function does not require solving the fixed point between equilibrium long-term bond prices and policy functions.

Moreover, if $W(\boldsymbol{b}, T)$ has a strict maximizer, that is, if the sovereign is not indifferent between two exit horizons, then that maximizer characterizes equilibrium prices and consumption policies. To see this, recall that in any equilibrium $\sup_T W(\boldsymbol{b}, T) = V(\boldsymbol{b}, \underline{V}^D) = W(\boldsymbol{b}, T(\boldsymbol{b}))$,

[13] To see this, suppose it did choose $b_{S,T} < \overline{B} - b_L$. It could increase consumption in period $T-1$ by a small amount. To satisfy its budget constraint, it increases $b_{S,T}$ by an equal amount. As long as this increase is less than $\overline{B} - b_L - b_{S,T} > 0$, the sovereign will exit the crisis zone on schedule. Note that in the period of exit, the sovereign is saving. In particular, it faces risk free rates and chooses $c = y - rb_L - (1+r)b_{S,T-1} + b_{S,T} < y - r(b_L + b_{S,T})$, where the last inequality follows from the requirement that $b_{S,T} < b_{S,T-1}$. The latter quantity is consumption while in the safe zone, which therefore has a lower marginal utility of consumption. Thus, shifting consumption into the crisis zone while maintaining the same time-until-exit improves welfare.

where the last expression uses the equilibrium time-until-exit associated with an equilibrium V. If the first expression has a unique maximizer, than there is a unique equilibrium time-until-exit $T(\boldsymbol{b})$. In what follows, we use the tie-breaking assumption that when indifferent, the sovereign exits sooner rather than later, and in this way, we can pin down unique equilibrium prices and consumption paths.

Before proceeding, we can now let $\boldsymbol{D} = \{\boldsymbol{b} \in \mathcal{B} | \sup_T W(\boldsymbol{b}, T) < \underline{V}^D\}$. This region defines the outer boundary of the equilibrium crisis zone. Note that $\underline{V}^D < \overline{V}^D$ and continuity of V implies the $\boldsymbol{ND}$ and $\boldsymbol{D}$ regions are always separated by a non-empty crisis region. We have now characterized the equilibrium value function over the entire state space:

Proposition 3. ***Equilibrium Value Functions:*** *In any Markov Perfect Equilibrium, we have:*

$$V(\boldsymbol{b}, V^D) = \begin{cases} \frac{u(y - r(b_S + b_L))}{1-\beta} \text{ if } \boldsymbol{b} \in \boldsymbol{ND} \text{ and all } V^D \\ \sup_{T \geq 1} W(\boldsymbol{b}, T) \text{ if } \boldsymbol{b} \in \boldsymbol{C} \text{ and } V^D = \underline{V}^D. \end{cases}$$

and where $V(\boldsymbol{b}, \underline{V}^D) < \underline{V}^D$ *for all* $\boldsymbol{b} \in \boldsymbol{D}$*, and* $V(\boldsymbol{b}, \overline{V}^D) < \overline{V}^D$ *for all* $\boldsymbol{b} \in \boldsymbol{C} \cup \boldsymbol{D}$.

With equilibrium value functions in hand, we now turn our attention to equilibrium bond prices.

3.4 Iso-T Regions and Iso-V Curves

Equilibrium bond prices can be characterized from the optimal time-until-exit. In Figure 2 we plot "iso-T" loci in the space of short- and long-term debt. That is, along each of the iso-T curves, the sovereign is indifferent between exiting in T and $T+1$ periods, while within each region there is a unique optimal exit time. Closest to the no-default region, we have $T(\boldsymbol{b}) = 1$, and as we add debt the time-until-exit increases. As implied by Claim 1, if the sovereign holds only long-term debt, $T(0, b_L) = \infty$ is always optimal, as there is no incentive to save. Thus the $T = \infty$ locus originates from the horizontal edge of the no-default zone. If $b_S = 0$, then either $T = 0$ or $T = \infty$.

We can characterize the slope of the equilibrium iso-T curves using the fact that along the (outer) boundaries of each region the sovereign is indifferent between exiting in T and $T+1$ periods. That is, $\{T, T+1\} \in \operatorname{argmax}_{\tilde{T}} W(\boldsymbol{b}, \tilde{T})$, and so $W(\boldsymbol{b}, T) = W(\boldsymbol{b}, T+1)$. Using the expression for C^T from (4), the implicit function theorem, and some algebra, we have:

$$-\left.\frac{db_S}{db_L}\right|_{W(\boldsymbol{b}, T(\boldsymbol{b})) = W(\boldsymbol{b}, T(\boldsymbol{b})+1)} > \frac{q_L(\boldsymbol{b})}{q_S(\boldsymbol{b})}, \tag{6}$$

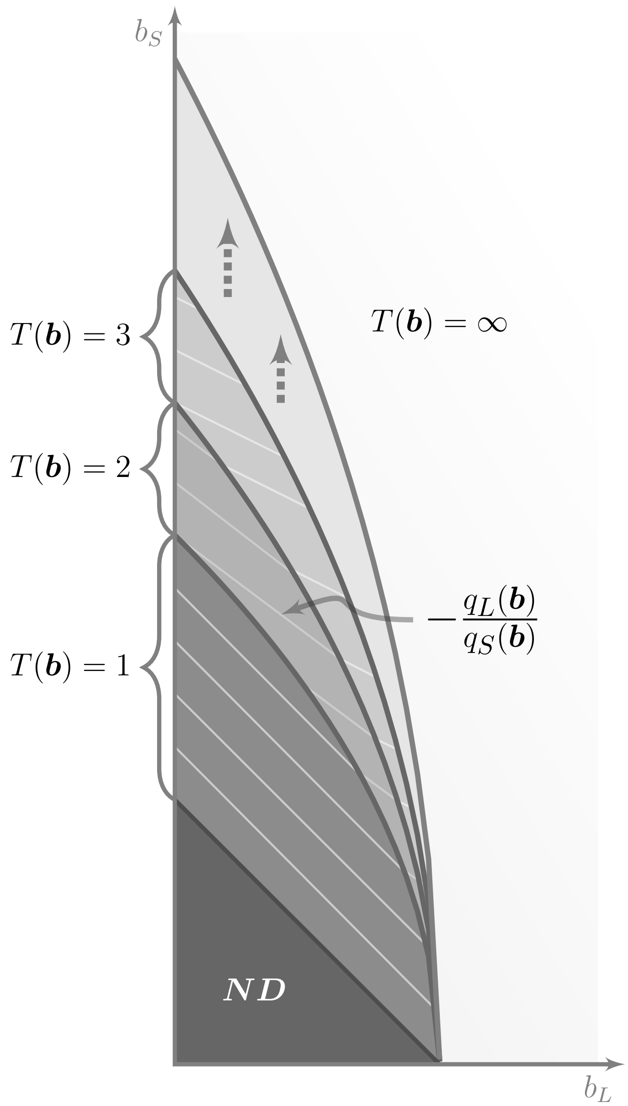

Figure 2: Equilibrium Iso-T Regions. Lighter shaded areas represent regions of higher equilibrium exit times. The vertical arrows represent the infinity of different Iso-T regions that lie in that area until reaching the boundary of $T(\boldsymbol{b}) = \infty$ region. The white linear segments represent prices in the different regions, with regions with higher exit times having flatter lines (that is, lower perpertuity prices).

where $T(\boldsymbol{b})$ is the optimal exit time. Recall that if the sovereign is indifferent between exiting in T and $T+1$, equilibrium prices assume that the sovereign exits in T periods. Therefore, the right-hand-side of (6) is the relative price of a long-term bond conditional on exiting in T periods.

The implication of (6) for Figure 2, is that as we increase b_S and decrease b_L along a line of constant market value $(q_S b_S + q_L b_L)$, the optimal exit time is weakly decreasing. The straight lines in Figure 2 have slope $-q_L/q_S$, and within an iso-T region depict lines of constant market value. As we cross the boundary of an iso-T region moving in the "northwest" direction, the exit time falls by one period, and the ratio of equilibrium prices q_L/q_S increases.

We can use the ratio q_L/q_S to construct "iso-V" curves as well; that is, loci of (b_S, b_L) such that $V(\boldsymbol{b})$ is constant. These are depicted in Figure 3. Because of the discreteness of a period, in the interior of an iso-T region, small movements in $\boldsymbol{b}$ will not change the optimal exit time, and hence do not change prices. Therefore W and V are differentiable in b_S and b_L in the interior of an iso-T region. Note that conditional on T, constant W implies a constant C^T, and differentiation of (4), and using the expression for q_L from equation (3), delivers:

$$\left.\frac{db_S}{db_L}\right|_{V(\boldsymbol{b})=V} = -\frac{q_L(\boldsymbol{b})}{q_S(\boldsymbol{b})}.$$

This slope reflects the fact that conditional on an equilibrium T, short-term and long-term bonds can be traded at these relative prices, holding constant total market value. As such (small) exchanges leave exit time unchanged, and market value represents expected payments to bondholders, the sovereign is indifferent along this iso-market value lines. However, as we shift the portfolio we eventually reach the boundary of the iso-T region. By definition, the sovereign is indifferent between exiting in T and $T+1$ periods at the boundary, and the iso-V line is continuous but shifts slope to reflect the new region's prices.

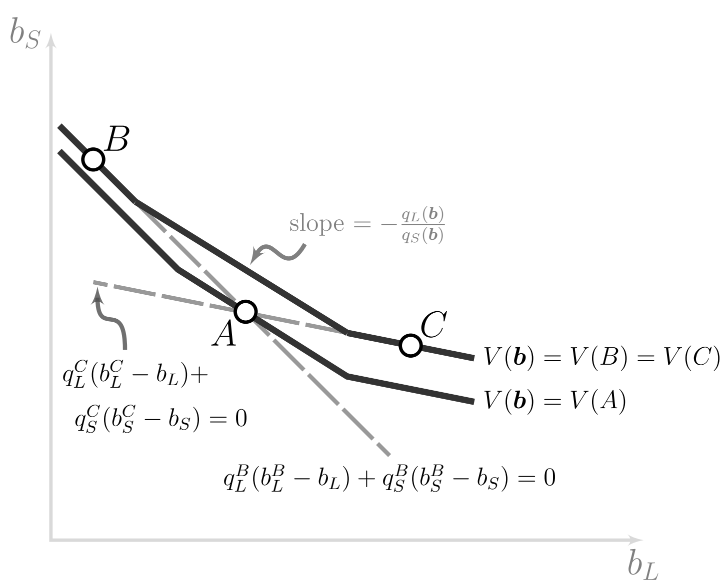

Figure 3: Equilibrium Iso-V curves and Zero-Cost Trades. Solid lines represent two distinct equilibrium iso-V curves. Points B and C represent possible zero costs trades from A.

Note that as we add debt, V decreases, and so the value function is quasi-convex in the crisis region. That is, a line connecting two points on an iso-V curve passes through regions of weakly lower values. This facet is somewhat masked in Figure 3 as the iso-V curves look like "standard" indifference curves from quasi-concave preferences. However, as debt is a "bad" rather than a "good," the iso-V lines are actually inverted relative to standard preferences. It is important to note that the underlying preferences for the sovereign are standard, concave preferences. The iso-V lines are *equilibrium* outcomes. As we move through the debt space, the sovereign is facing different incentives to save (and hence different prices), which generates the quasi-convexity of the value function in equilibrium. The iso-V curves also help clarify that the sovereign is indifferent between transition paths that involve the same sequence of iso-V curves; that is, there is a local indeterminacy regarding portfolio choice due to the discreteness of time. The sovereign moves through the iso-T regions in consecutive fashion (T, $T-1$, ...), but within each iso-T region there is a line segment of constant market value of debt along which the sovereign is indifferent.

The Yield Curve

Before moving on, let us discuss the implications of the results so far for the yield curve. This is of particular interest as the yield curve inverts in emerging markets when bond spreads are high (Broner et al., 2013). In our model, such an inversion occurs naturally, reflecting that deleveraging implies the long-run risk of default is less than the short-run risk.

To see this formally, let r_S and r_L denote the effective yields for the short and the long term bond, respectively. The yields can be written as:

$$r_S = \frac{R}{q_S} - 1, \qquad \text{and} \qquad r_L = \frac{r}{q_L}.$$

Note that when $\boldsymbol{b} \in \boldsymbol{ND}$, $q_S = q_L = 1$, and both yields are equal to r, as expected (that is, the yield curve is flat at risk-free yields). When $\boldsymbol{b} \in \boldsymbol{C}$, $q_S = 1 - \lambda$ and

$$r_S - r_L = \frac{r + \lambda}{1 - \lambda} - \frac{r}{q_L}.$$

For $T(b) = \infty$, $q_L = \frac{(1-\lambda)r}{r+\lambda}$ and thus the above implies $r_S = r_L$. This reflects that if the sovereign never exits $\boldsymbol{C}$, long-run risks and short-run risks are equivalent. However, if $T(b) < \infty$, then there is a chance the government exits $\boldsymbol{C}$ in finite time, and therefore long-run risk is lower than short-run risk. To see that this implies $r_S > r_L$, note that $q_L > \frac{(1-\lambda)r}{r+\lambda}$ for $T(b) < \infty$. Substituting into the above expression implies that $r_S - r_L > 0$.

3.5 Zero-Cost Trades and Optimal Portfolio Management

Theorem 1 stated that a policy of passive long-term debt and active short-term debt management is consistent with optimality. This allows us to characterize any equilibrium by using Problem PW. However, while Theorem 1 states it suffices to use short-term bonds, it does not necessarily rule out an equally optimal alternative that involves long-term bonds. Under commitment, short-term and long-term bonds span each other in our environment, and the portfolio would be indeterminate (as it is in the no-default region). However, this indeterminacy does not survive the absence of commitment. We now explain why limited commitment implies that the use of long-term bonds may be strictly sub-optimal in equilibrium. In particular, we show that debt dynamics that involve active long-term bond policies *shrink* the budget set at equilibrium prices.

We can use Figure 3 to discuss this. To do so, we first consider "zero-cost trades." These are portfolio shifts that require and generate no net payments. Specifically, suppose the state at the end of a period is $\boldsymbol{b}$, and the sovereign decides to engage in another round of trading, moving to a new state $\boldsymbol{b}'$ such that:

$$q_L(\boldsymbol{b}')\,(b'_L - b_L) + q_S(\boldsymbol{b}')\,(b'_S - b_S) = 0. \tag{7}$$

Note that the prices are those of the new portfolio ($\boldsymbol{b}'$), consistent with the equilibrium going forward from $\boldsymbol{b}'$. We claim the following:

Lemma 3. *Suppose that $\boldsymbol{b}$ and $\boldsymbol{b}'$ satisfy the zero-cost trade equation (7). If $T(\boldsymbol{b}') \notin \operatorname{argmax}_T W(\boldsymbol{b}, T)$, then $V(\boldsymbol{b}') < V(\boldsymbol{b})$. That is, if the equilibrium exit time at $\boldsymbol{b}'$ is not optimal for the sovereign at $\boldsymbol{b}$, the zero-cost trade is strictly welfare reducing.*

This lemma implies that any zero-cost trade that changes optimal exit time (and hence long-term bond prices) is welfare reducing for the sovereign. Note that in a zero-cost trade, the sovereign is actively trading long-term bonds. Importantly, the result is independent of the sign of $b'_L - b_L$; that is, whether the sovereign is selling or buying its long-term bonds, such trades are welfare reducing if they change the exit time.

The formal proof is in the appendix (and follows straightforwardly from the fact that $T(\boldsymbol{b})$ is chosen optimally), but the result can be discussed intuitively using Figure 3. Let point $\boldsymbol{b} = A$ represent an initial state. Now consider a zero cost trade that buys long-term bonds in exchange for short-term bonds. Such a move will necessary move the economy to a point such as $\boldsymbol{b}' = B$, at which the exit time is lower. The line connecting A to B contains the possible zero-cost trades under point-B prices. By construction, this line has slope $-q_L(B)/q_S(B)$, which makes it tangent to the iso-V line at B. Moreover, because B is in a lower exit-time region, this slope is steeper than the indifference curve at A. This implies that B is on a lower iso-V curve than A. The intuition is that long-term bond prices are relatively high at B, and buying back the long-term bonds involves a net transfer to bondholders via a capital gain. The transfer arises from the greater incentives to exit quickly at point B.[14]

Similarly, point C is a zero-cost trade from A in which the sovereign issues long-term bonds and buys back short-term bonds. Again, the line connecting A to C has slope $-q_L(C)/q_S(C)$, the equilibrium relative price at C. As C is in a region with longer time-until-exit than A, the relative price of long-term bonds is lower at C. This implies the trading line is shallower than the iso-V curve at A. This trade also leads to a lower iso-V line. In this case, the sovereign is issuing long-term bonds at low prices. The existing bond holders do take a capital loss, but the sovereign is no better off as the low price on the newly issued bonds implies it can retire very little of the short-bonds (hence the shallow slope of the zero-cost line to point C).

Lemma 3 concerns zero-cost trades, but the result helps evaluate the benefits of any portfolio shift, including those involving a reduction in total debt. In particular, any trade from state $\boldsymbol{b}$ to another state $\boldsymbol{b}'$ can conceptually be decomposed into trading only short-term bonds and then shuffling the portfolio in a zero-cost trade.

To see this, consider the original budget constraint in a period in which the sovereign enters with $\boldsymbol{b}$ and exits with $\boldsymbol{b}'$, conditional on no change in V^D:

$$c + (1+r)b_S + rb_L = y + q_S(\boldsymbol{b}')b'_S + q_L(\boldsymbol{b}')(b'_L - b_L). \tag{8}$$

This situation is shown by the solid arrow in Figure 4.

For our conceptual decomposition, let us define b''_S as follows:

$$q_L(\boldsymbol{b}')(b'_L - b_L) + q_S(\boldsymbol{b}')(b'_S - b''_S) = 0. \tag{9}$$

That is, $\boldsymbol{b}'' \equiv (b''_S, b_L)$ is such that $\boldsymbol{b}'$ represents a zero cost trade from $\boldsymbol{b}''$, at $q(\boldsymbol{b}')$ prices. We can then rewrite the budgetary impact of moving from $\boldsymbol{b}$ to $\boldsymbol{b}'$ by conceptually breaking up the movement into two trades within the period: (i) the sovereign moves to $\boldsymbol{b}'' = (b''_S, b_L)$ by paying down (or issuing) short-term bonds and not trading long-term bonds, and then immediately makes (ii) a zero-cost trade from $\boldsymbol{b}''$ to $\boldsymbol{b}'$.

[14] This is reminiscent of the buyback "boondoggle" of Bulow and Rogoff (1988), although the mechanism is via incentives in this case, while it is through liquidation value in the Bulow-Rogoff environment. In fact, in the Bulow-Rogoff environment, the sovereign would be better off issuing long-term bonds to dilute exiting bondholders' liquidation claims. This is not the case in our environment, as trades to point C in Figure 3 demonstrate.

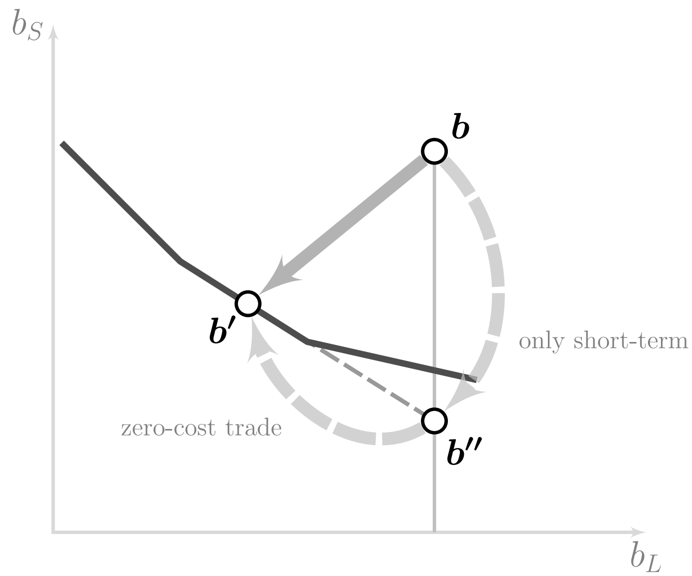

Figure 4: A Decomposition. A movement from $\boldsymbol{b}$ to $\boldsymbol{b'}$ can be decomposed as a trade from $\boldsymbol{b}$ to $\boldsymbol{b''}$ using only short-term debt, and a zero-cost trade from $\boldsymbol{b''}$ to $\boldsymbol{b'}$. The utility at $\boldsymbol{b''}$ is strictly higher than at $\boldsymbol{b'}$.

Note that, using equations (8) and (9), the budget constraint can be rewritten as:

$$c + (1+r)b_S + rb_L = y + q_S(\boldsymbol{b}')b''_S. \tag{10}$$

Now consider an alternative policy of just trading to $\boldsymbol{b}''$, rather than to $\boldsymbol{b}'$. From Lemma 3, we know that the continuation value at $\boldsymbol{b}''$ is weakly greater than $\boldsymbol{b}'$ as the two points are related by a zero-cost trade. Thus trading to $\boldsymbol{b}''$ weakly dominates trading to $\boldsymbol{b}'$ if it does not require lower current-period consumption. The required consumption for the trade to $\boldsymbol{b}''$ is given by equation (10) with $q_S(\boldsymbol{b}')$ replaced by $q_S(\boldsymbol{b}'')$. But note that the weakly greater equilibrium value at $\boldsymbol{b}''$ also implies that $q_S(\boldsymbol{b}'') \geq q_S(\boldsymbol{b}')$, with strict inequality if $\boldsymbol{b}'' \in \boldsymbol{ND}$. Relative to (10), this weakly relaxes the current-period budget constraint, and thus the required consumption to trade to $\boldsymbol{b}''$ is higher.[15]

Hence, the sovereign is weakly worse off trading to $\boldsymbol{b}'$ than to simply trade to $\boldsymbol{b}''$ and stay put, and strictly better off if moving to $\boldsymbol{b}'$ involves different incentives to exit than $\boldsymbol{b}''$. Thus an exit strategy that involves only short-term bonds weakly dominates alternative paths, and strictly dominates those that involve changes in long-term bond prices. In a situation of deleveraging under the risk of default, a sovereign will shy away from using the long-term bond market: at equilibrium prices, it is too expensive to use it.

[15] More precisely, this relaxes the budget constraint as long as $b''_S \geq 0$. However, if the implied $b''_S < 0$, then, from Claim 1, it is optimal to just remain at $\boldsymbol{b}$.

We summarize the results of this subsection in the following theorem:[16]

Theorem 2. *Let $\boldsymbol{b} \in \boldsymbol{C}$ and suppose the sovereign decides to exit the crisis region in T periods. Relative to the short-term-bond-only strategy that solves problem $W(\boldsymbol{b}, T)$, any alternative portfolio strategy starting from $\boldsymbol{b}$ that involves active long-term bond management is weakly sub-optimal, and generically strictly sub-optimal if the equilibrium bond prices along the transition differ from those along the W trajectory.*

4 Maturity and Rollover Crises

In the preceding analysis, we were able to uniquely characterize equilibrium regions and value functions. Moreover, bond prices were uniquely determined subject to the tie-breaking assumption when the sovereign is indifferent over two exit times. Conditional on an exit time, there is a local indeterminacy about portfolios, as small changes in maturity structure may not induce a discrete change in exit time. This local indeterminacy aside, the fact that the equilibrium is pinned down leaves no room for coordination failures and the resulting self-fulfilling rollover crises. In this section we slightly modify the environment to allow for such possibilities.

Given our focus on the choice of the maturity structure, this modification is an important one, as it has been pointed out (see, for example Cole and Kehoe, 2000) that a longer maturity structure mitigates the vulnerability to a self-fulfilling rollover crisis. In this section, we consider how this insight applies to a model with active maturity management. We show that the main result of the preceding analysis goes through; that is, long-term bonds are not used during deleveraging, save for an extension of the maturities at the end of the process. That is, the desired lengthening of maturities is delayed as long as possible in order to minimize the costs of deleveraging.

As noted before, we need to alter the environment to allow coordination failures by the lenders. To this end, we alter the timing within a period (see Figure 5). The start of the period is the same as the benchmark. Specifically, assuming no default in a previous period, the sovereign enters the period with $\boldsymbol{b}$ and draws V^D. Before making a default decision, the sovereign auctions newly issued bonds. The next step is bond-market settlement, which involves the sovereign's decision to repay outstanding claims or not. If the sovereign does not default, it settles outstanding claims using a combination of endowment y and newly raised money.[17] In the no-default case, the environment is identical as in the benchmark.

[16]The term "generically" in the proposition refers to the possibility that the sovereign is indifferent between exiting in T and $T+1$ periods.

[17]See Lorenzoni and Werning (2013) for an alternative source of multiplicity along the lines of Calvo (1988).

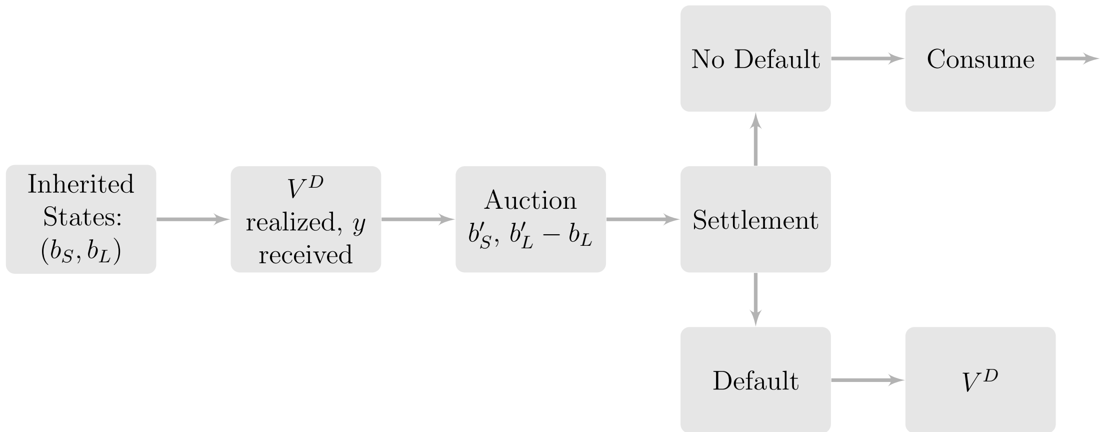

Figure 5: Timing within a Period in the Model with Coordination Failures. Note the difference with Figure 1.

However, if the sovereign defaults during settlement, the money raised at auction is distributed across *all* bond holders based on their respective positions. In particular, the money raised from new bond holders is partially transferred to existing bond holders.[18] Importantly, this exposes newly issued bonds to default risk within the period of issuance, and, as we shall see, opens the door to rollover crisis.[19]

Given the new timing, we need to update the break-even conditions (BE) used to price bonds in the benchmark environment. Specifically, the fact that contemporaneous default is a possibility implies that:

$$\begin{aligned} q_S(\boldsymbol{b}, V^D, \boldsymbol{b'}) &= (1 - D(\boldsymbol{b}, V^D, \boldsymbol{b'}))\mathbb{E}\left[1 - D(\boldsymbol{b'}, V^{D\prime}, \boldsymbol{b''}) \middle| V^D\right] \\ q_L(\boldsymbol{b}, V^D, \boldsymbol{b'}) &= (1 - D(\boldsymbol{b}, V^D, \boldsymbol{b'}))\mathbb{E}\left[\left(1 - D(\boldsymbol{b'}, V^{D\prime}, \boldsymbol{b''})\right)\left(\frac{r + q'_L}{1 + r}\right) \middle| V^D\right], \end{aligned} \tag{BE$'$}$$

where q'_L as before is the long-term bond price next period given the government's policy functions. The important difference between (BE) and (BE$'$) is that the current period's $\boldsymbol{b}$ is a relevant state variable in the pricing equation. This reflects that the initial debt is relevant for the sovereign's current-period default decision, and this decision is relevant for

[18] This is consistent with the fact that all bondholders have equal seniority in the event of default.

[19] If money raised from newly issued bonds were instead returned in entirety to the new bond holders, the environment would be equivalent to the benchmark. Note also that the environment is a slight departure from that of Cole and Kehoe (2000). Cole and Kehoe differ from that of Figure 5 in that in the event of default, the sovereign gets to keep the proceeds from the newly issued bonds. This requires keeping track of off-equilibrium outcomes in which the sovereign issues a large amount of new bonds and consumes the proceeds while defaulting, an unnecessary complication. The critical assumption is that new bond holders are immediately at risk, not who receives the money.

newly issued bonds.[20]

We now consider equilibria in which a rollover crisis shrinks the area of no default, leaving the country vulnerable to default in regions of the state space that would otherwise be risk free.

4.1 Coordination Failures

The definition of equilibria in this Rollover Crisis Environment is the same as the benchmark (Definition 1), with (BE′) replacing (BE) in condition (iii). With the new timing, we no longer have uniqueness in terms of equilibrium regions, values, or prices. However, it is straightforward to verify that the benchmark equilibrium remains an equilibrium under the new timing.[21] We shall denote the benchmark equilibrium's value function and prices as $V^\star$ and $q_j^\star$, $j = S, L$, respectively, and use them as references for the equilibrium with rollover crises characterized in this section.

As in the benchmark we divide the state space into three regions: a no-default region $\tilde{\boldsymbol{ND}}$, a crisis region $\tilde{\boldsymbol{C}}$, and an immediate default region $\tilde{\boldsymbol{D}}$, where throughout the section a tilde indicates a departure from the benchmark. The benchmark $\boldsymbol{ND}$ is the largest possible no-default region. We now construct an equilibrium with the smallest possible no-default zone:

$$\tilde{\boldsymbol{ND}} = \left\{ \boldsymbol{b} \in \mathcal{B} \,\middle|\, u(y - (1+r)b_S - rb_L) + \beta \frac{u(y - rb_L)}{1-\beta} \geq \overline{V}^D \right\}. \tag{11}$$

To see how this region arises in equilibrium, suppose that creditors refuse to roll over maturing bonds as they anticipate a default. The condition in (11) states that even if creditors refuse to roll over maturing bonds and $V^D = \overline{V}^D$, the sovereign would still prefer to repay at all periods rather than to default. We contrast $\tilde{\boldsymbol{ND}}$ with the benchmark $\boldsymbol{ND}$ in Figure 6. The region $\tilde{\boldsymbol{ND}}$ is asymmetric with respect to maturity, with the boundary favoring long-term debt. This reflects that only short-term debt is vulnerable to a roll over crisis. The boundary of the benchmark $\boldsymbol{ND}$ is the dashed line with slope -1, the ratio of risk-free prices. Note that $\tilde{\boldsymbol{ND}}$ is a strict subset of the benchmark $\boldsymbol{ND}$, with the boundaries overlapping only at $b_S = 0$ (at which point the $\boldsymbol{ND}$ region is tangent to $\tilde{\boldsymbol{ND}}$).

To clarify, we are constructing an alternative equilibrium with a smaller no-default region $\tilde{\boldsymbol{ND}}$. Within the equilibrium, there is no uncertainty about the strategies of bondholders. In particular, we are not introducing a sunspot such that agents switch from one equilibrium strategy to the other. The equilibrium features a coordination failure that generates a

[20] The break-even condition should allow for money raised in new bond issuances to be rebated to all bondholders; since this is always zero in equilibrium, we omit it from the expressions.

[21] We omit the details of this argument in order to focus on equilibria with coordination failures.

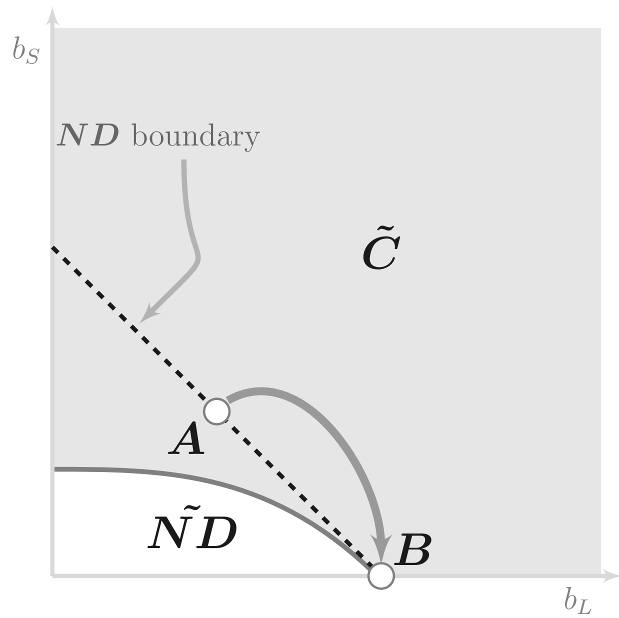

Figure 6: No-Default Region with Rollover Crises. Shaded area represents the $\tilde{C}$ region. In equilibrium, there would be an immediate trade from any point such as A to point B, at risk free prices. Note that point B lies in the boundary of $\boldsymbol{ND}$, this is important for the equivalence result of Proposition 4.

no-default region that is strictly smaller than $\boldsymbol{ND}$.

Outside $\tilde{\boldsymbol{ND}}$, the sovereign prefers to default when creditors are unwilling to roll over maturing bonds. In particular, suppose the sovereign faces the following price schedules in the weak enforcement regime:

$$q_j(\boldsymbol{b}, \overline{V}^D, \boldsymbol{b'}) = \begin{cases} 1 & \text{if } \boldsymbol{b} \in \tilde{\boldsymbol{ND}}, \boldsymbol{b'} \in \tilde{\boldsymbol{ND}} \\ p(\boldsymbol{b}, b'_S + b'_L) \in (0,1) & \text{if } \boldsymbol{b} \notin \tilde{\boldsymbol{ND}}, \boldsymbol{b'} \in \tilde{\boldsymbol{ND}}, \text{ and } b'_S + b'_L < B^p(\boldsymbol{b}) \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

for $j = S, L$, $\boldsymbol{b'} = (b'_S, b'_L)$, and $\boldsymbol{b}, \boldsymbol{b'} \in \mathcal{B}$. The first line reflects that bonds can be rolled over at risk free prices if the initial $\boldsymbol{b} \in \tilde{\boldsymbol{ND}}$. The second line defines "off equilibrium" prices that rule out the government repurchasing long-term bonds at price zero and not defaulting, which is inconsistent with equilibrium. The functions p and B^p ensure prices are consistent with equilibrium over the entire debt state space, even for outcomes $(\boldsymbol{b}, \boldsymbol{b'})$ never chosen in equilibrium.[22] The final line of (12) reflects that for $\boldsymbol{b} \notin \tilde{\boldsymbol{ND}}$, the government is not

[22]In particular, suppose the government could repurchase bonds at price zero to a point $B' = b'_S + b'_L$, where $(b'_S, b'_L) \in \tilde{\boldsymbol{ND}}$. For this to be an equilibrium price, we require $u(y - Rb_S - rb_L) + \beta u(y - rB')/(1-\beta) \leq \overline{V}^D$.

afforded the opportunity to fully roll over its debt if $V^D = \overline{V}^D$, and thus must choose between immediately paying down its debt or default.

Faced with these prices, the sovereign's value function in the weak enforcement state conditional on no-default this period is:

$$V(\boldsymbol{b}, \overline{V}^D) = \frac{u(y - r(b_S + b_L))}{1-\beta} \text{ if } \boldsymbol{b} \in \tilde{\boldsymbol{ND}} \tag{13}$$

and $V(\boldsymbol{b}, \overline{V}^D) \leq \overline{V}^D$ if $\boldsymbol{b} \notin \tilde{\boldsymbol{ND}}$.

With prices and values determined for the weak enforcement regime, we now turn to the strong enforcement regime. The main result of this section is that the prices and value functions in the crisis equilibrium conditional on being in the strong enforcement regime are the same as the benchmark without rollover crises:

Proposition 4. *There exists a Markov Perfect Equilibrium of the Rollover Crisis Environment that has the following characteristics: Let* $\tilde{\boldsymbol{ND}}$ *be defined by* (11); $\tilde{\boldsymbol{D}} = \boldsymbol{D}$ *be the default zone as in the benchmark model; and* $\tilde{\boldsymbol{C}}$ *the remaining region of* $\mathcal{B}$*. The equilibrium value function for* $V^D = \overline{V}^D$ *is given by* (13)*; and for* $V^D = \underline{V}^D$ *satisfies* $V(\boldsymbol{b}, \underline{V}^D) = V^\star(\boldsymbol{b}, \underline{V}^D)$ *for all* $\boldsymbol{b} \in \mathcal{B}$*, where* $V^\star$ *is the value function from the benchmark equilibrium.*

This proposition states that the sovereign's value is the same as in the no-rollover-crisis benchmark, conditional on $V^D = \underline{V}^D$. This is so despite the presence of rollover risk, which is manifested in a strictly smaller no-default region and strictly larger crisis region. To understand why the two environments are equivalent in the strong enforcement regime, consider a point $\boldsymbol{b} \in \boldsymbol{ND}$ but not in $\tilde{\boldsymbol{ND}}$. This is the region in which a "self-fulfilling crisis" induces default in the current environment, but not in the benchmark. However, absent the onset of the run this period (that is, if the current $V^D = \underline{V}^D$), the sovereign can still trade sovereign bonds. In particular, it can trade *into* $\tilde{\boldsymbol{ND}}$ at risk-free prices. More generally:

Proposition 5. *The prices associated with the equilibrium of Proposition 4 are given by* (12)

The threshold $B^p(\boldsymbol{b})$ is the threshold for B' such that this holds with equality, and so zero is an equilibrium price for any $B' \geq B^p(\boldsymbol{b})$. For $B' < B^p(\boldsymbol{b})$, $p(\boldsymbol{b}, B')$ is constructed so that the sovereign is strictly indifferent between default and not. That is $p(\boldsymbol{b}, B') = \tilde{p}$, where $\tilde{p} \in (0,1)$ solves $u(y - Rb_S - rb_L + \tilde{p}(B' - b_L)) + \beta u(y - rB')/(1-\beta) = \overline{V}^D$, and so the government is indifferent to default and bringing total debt down to B' at price $p(\boldsymbol{b}, B')$. This indifference is then used to construct a mixing equilibrium, where the sovereign defaults with probability $1 - p(\boldsymbol{b}, B')$, justifying the price.

for $V^D = \overline{V}^D$ *and for* $V^D = \underline{V}^D$ *and* $\boldsymbol{b} \notin \boldsymbol{D}$ *are:*

$$q_S(\boldsymbol{b}, \underline{V}^D, \boldsymbol{b'}) = \begin{cases} 1 \text{ } \textit{if} \text{ } \boldsymbol{b'} \in \tilde{\boldsymbol{ND}} \\ 1 - \lambda \text{ } \textit{if} \text{ } \boldsymbol{b'} \in \tilde{\boldsymbol{C}} \\ 0 \text{ } \textit{otherwise}; \end{cases} \tag{14}$$

and

$$q_L(\boldsymbol{b}, \underline{V}^D, \boldsymbol{b'}) = \begin{cases} 1 \text{ } \textit{if} \text{ } \boldsymbol{b'} \in \tilde{\boldsymbol{ND}} \\ 1 - \lambda \text{ } \textit{if} \text{ } \boldsymbol{b'} \in \boldsymbol{ND} - \tilde{\boldsymbol{ND}} \\ q_L^\star(\underline{V}^D, \boldsymbol{b'}) \text{ } \textit{otherwise}, \end{cases} \tag{15}$$

where $q_L^\star$ *is the price function of the benchmark equilibrium.*

This proposition says that if the end-of-period debt places the sovereign in the no-default region, bonds carry risk-free prices absent a run. Thus, in the strong enforcement regime, the sovereign can always trade into the no-default region. More importantly, such trades are zero-cost trades if $\boldsymbol{b} \in \boldsymbol{ND}$. To see this, suppose that $\boldsymbol{b} = (b_S, b_L)$ is such that $b_S + b_L = \overline{B}$; that is, $\boldsymbol{b}$ is on the boundary of the benchmark no-default region. This is depicted in Figure 6 by point as A. The cost to trade to point $B = (0, \overline{B}) \in \tilde{\boldsymbol{ND}}$ from A is:

$$\begin{aligned} & q_S(\boldsymbol{b}, \underline{V}^D, B)(0 - b_S) + q_L(\boldsymbol{b}, \underline{V}^D, B)(\overline{B} - b_L) \\ & = -b_S + \overline{B} - b_L = 0. \end{aligned}$$

Note a key distinction between this zero-cost trade and those considered in the benchmark. In the benchmark, the boundary of the no-default region had a slope of -1, which is the ratio of risk-free prices. Thus it was never possible to trade into the no-default region at these prices. However, in the current environment the no-default region is not a line of slope -1, and thus it is possible that a zero-cost trade at risk-free prices can move the sovereign into the no-default region. In fact, it is optimal for the sovereign to trade into the no-default region $\tilde{\boldsymbol{ND}}$ as soon as it reaches the boundary of the benchmark no-default region. The value at this boundary is therefore the same as in the benchmark. Moreover, the prices outside the benchmark $\boldsymbol{ND}$ are the same as in the benchmark. This allows the sovereign to pursue the same policies and achieve the same welfare as in the benchmark case without roll over crises.

The main distinction between the benchmark and the current environment with crises is what happens at the boundary of $\boldsymbol{ND}$. In the benchmark, the sovereign could simply remain at the boundary, regardless of its maturity structure. In the current environment, the sovereign must lengthen maturities in order to reach the new no-default region $\tilde{\boldsymbol{ND}}$. This reflects the asymmetry of maturity choice in the presence of rollover risk. However, prior to this reshuffling of maturity at the boundary, the sovereign's incentives not to trade in long-term debt markets remain the same as in the benchmark. The optimal sequencing is therefore to remain passive in long-term debt markets while deleveraging, and then lengthen maturity at the end of the process.

From the preceding analysis, it is clear that the model easily extends to arbitrary no-default regions. Whatever the shape of this region in equilibrium, the relevant boundary is at what point in the debt space can a zero-cost trade reach into the no-default region. Absent a contemporaneous default, the sovereign will find it optimal to exploit such a trade. In terms of welfare and risk, therefore, this extended boundary is the relevant frontier of the no-default region and this boundary by definition is symmetric with a slope of -1.

5 Efficient Restructuring

The analysis thus far has concerned equilibrium debt management. That is, faced with competitive prices, the sovereign optimally chooses its maturity structure. We now turn to the question of whether there is a potential Pareto improvement in a non-competitive setting. That is, if creditors and the sovereign could bargain over a debt restructuring, what would be an efficient outcome of that process? Specifically, we consider a one-shot restructuring, after which the sovereign and creditors resume equilibrium behavior.[23]

To answer this, we can appeal to Hopenhayn and Werning (2008), who derive a general implementation result for the efficient contract between a lender and borrower when the borrower is hit by outside option shocks. In particular, they find that (in an environment without coordination failures) an efficient allocation can be implemented with contracts where the entire stock of debt (in all periods) is of one-period maturity. To discuss this result, we will restrict attention to the benchmark equilbrium,[24] and turn to the iso-V curves introduced in Figure 3, which we reproduce in Figure 7. We assume the initial state is in the crisis region, as there is no incentive to restructure in the no-default region. We hold off on the default region until after discussing restructuring in the crisis region.

The initial state is depicted by point $A = (b_S^A, b_L^A)$ along with its associated iso-V curve.

[23]Hatchondo et al. (2013) study the introduction of debt exchanges in a quantitative sovereign debt model with a long-term bond. They show that there are situations where a write-down of the debt can generate a Pareto improvement ex-post. Differently from us, they abstract from maturity choice.

[24]Given the equivalence result of the previous section, what follows will also work for the equilibrium with coordination failures, except that the efficient allocation will require the issuances of long-term bonds during the final exit from the crisis region.

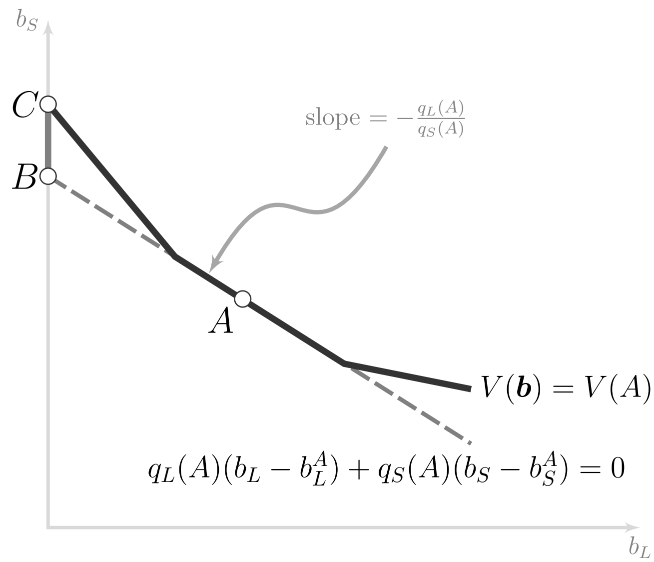

Figure 7: Equilibrium Iso-V Regions and Restructuring. The line segment from points B to C represents Pareto improvements over point A, that cannot be achieved as a market outcome.

The dashed line tangent to point A has a slope equal to the ratio prices evaluated at the initial state: $-q_L(A)/q_S(A)$, where we suppress $V^D = \underline{V}^D$ in the price notation. Along the dashed line, $q_S(A)(b_S - b_S^A) + q_L(A)(b_L - b_L^A) = 0$, and so the total amount of debt *evaluated at point-A prices* is constant. Of course, point-A prices are not equilibrium prices along the entire dashed line, as the equilibrium price of the long-term bond is sensitive to maturity structure. However, at point B, where the dashed line intersects the vertical axis ($b_L = 0$), the equilibrium market value of debt is the same as at point A. This is because $b_L = 0$ and $q_S(A) = q_S(B) = 1 - \lambda$. Therefore, creditors are indifferent between point A and point B at equilibrium prices. However, point B is a strict welfare improvement for the sovereign.

Similarly, we can follow the iso-V curve from point A until it also crosses the vertical axis in Figure 7 at point C. By definition, the sovereign is indifferent between point A and C. However, the market value of debt is greater at point C. This can be seen by recalling that the market value of debt is the same at A and B, and noting that point C has more debt (at the same short-term bond prices) as point B.

Therefore, a restructuring to the interval $\overline{BC}$ is a constrained Pareto improvement from point A. The intuition for why this improves on the equilibrium outcome is that it provides greater incentives for the sovereign to exit the crisis region. This helps mitigate the lack of commitment to pursue deleveraging in an expeditious manner. Moreover, this improvement cannot be implemented without coordination across creditors. In particular, long-term bondholders would like to hold out if bonds are restructured to shorter maturities. This involves a capital gain for long-term bondholders given the greater incentives to exit the crisis region after the restructuring. Note as well that there can be no improvement involving lengthening the maturity, as this will involve a drop in the market value of debt. In particular, lengthening maturity leads to a capital loss for bondholders after the restructuring, and so points to the right of A along the dashed line are not sustainable in equilibrium. Given the local indeterminacy, the constrained Pareto frontier extends from the vertical axis into the region with positive long-term debt; in particular, the extended Pareto frontier is the set of iso-T regions that contain the vertical axis.

We can extend the preceding analysis into the default zone. In the default zone, the sovereign's payoff is $\underline{V}^D$ (continuing to assume the strong-enforcement regime). The sovereign is therefore indifferent between any point in the default zone and the boundary between $\boldsymbol{D}$ and $\boldsymbol{C}$. Bondholders get zero in the default zone, and therefore are strictly better off moving to the boundary of the crisis zone. The above analysis then implies that the Pareto efficient restructuring in the default zone, $\boldsymbol{D}$, adjusts debt to where the boundary of the default region intersects the vertical axis. A similar point applies when the state is $V^D = \overline{V}^D$. The sovereign will default in the crisis region in the weak enforcement regime, and therefore efficient restructuring will reach the boundary of the no-default region. Along this boundary, maturity is irrelevant, and thus any point is an efficient outcome of restructuring.

The analysis of this section implies that efficient restructuring will shorten maturity. This provides the greatest incentives for the sovereign to deleverage and minimizes the length of time the sovereign and creditors are exposed to default risk. Lengthening of maturities in empirical debt restructurings are often motivated by providing "breathing room" for the sovereign. Within the context of the model with rollover crises, there is a point where lengthening of maturities is efficient and does mitigate rollover risk, but it may be better to delay such lengthening if another goal of the restructuring process is to induce the sovereign to deleverage. Moreover, the lengthening of maturity to mitigate rollover risk can be implemented at competitive equilibrium prices. Restructurings involving official lenders (like the IMF) also involve conditions on fiscal policy going forward. There is an issue of how enforceable such conditionality is, particularly as the official lender also lacks commitment to punish the debtor. This issue does not arise in the competitive equilibrium we consider. In particular, bond holders only demand to break even on average, which is always time consistent. This provides the incentive for the sovereign to reduce short-term debt as quickly as possible.

6 A More General Portfolio of Maturities

The previous results were obtained in a set up with just two assets: a one period bond and a perpetuity. In this section we show that the results can be extended to a more general portfolio of maturities.

We use the benchmark timing stated in Section 3 and reuse the notation from that section. We will now consider the case where we have $N+1$ bonds: a one-period bond plus N bonds of longer maturity. For tractability, we shall use the "random maturity" structure of Leland (1994a) and Chatterjee and Eyigungor (2012), which is also equivalent to the geometrically declining coupon formulation of Hatchondo and Martinez (2009).[25] Specifically, each of the bonds is indexed by $n \in \{0, 1, 2..., N\}$ and matures with *iid* probability $\delta_n \in [0, 1]$ each period. The one-period bond ($n = 0$) has $\delta_0 = 1$, and if there is a perpetuity it has $\delta = 0$; intermediate maturities set $\delta_n \in (0, 1)$, and have an expected maturity of $1/\delta_n$. To normalize all bond prices to one in a risk-free environment, bonds of type n promise a coupon of $r + \delta_n$ each period up to and including the date of maturity. For each bond of type n, there is a continuum of independent maturity realizations, and we appeal to the law of large numbers to state that a constant fraction δ_n of outstanding bonds b_n matures each period.

Absent default this period, the budget constraint of the sovereign is:

$$c + \sum_{n\in\{0,1,...,N\}} (r + \delta_n)b_n \leq y + \sum_{n\in\{0,1,...,N\}} q_n(V^D, \boldsymbol{b}')(b'_n - (1 - \delta_n)b_n)$$

where $\boldsymbol{b} = (b_0, b_1, ..b_N)$, and q_n represents the equilibrium price schedule of bond n. The term on the right represents net new issuances of each type of bond n, adjusting for the fraction δ_n that matures that period.

Letting D denote the default policy, it follows that the break-even conditions for the foreigners imply that:

$$q_n(V^D, \boldsymbol{b}') = \mathbb{E}\left[(1 - D(\boldsymbol{b}', V^{D\prime}))\left(\frac{r + \delta_n + (1 - \delta_n)q'_n}{1 + r}\right)\right]$$

where q'_n denotes the equilibrium price of bond n the following period.[26] As before, we impose that $q_n(\boldsymbol{b}, V^D, \boldsymbol{b}') \in [0, 1]$ for all i and states to rule out bubbles. Also as before, we restrict attention to $b_n \geq 0$ for all $n \in \{0, 1, ..., N\}$, and redefine $\mathcal{B} \equiv [0, 1)^{N+1}$.

Denoting by B_n the policy function for bond n, we can define a Markov Perfect Equilibrium

[25] While the random maturity formulation is tractable, our results hold for any bond portfolio with non-contingent payment terms, as long as there exits a one-period bond.

[26] Where we input δ_n for the expected probability that the bond matures next period.

in the same manner we did in Section 3. We can divide the state space into no-default region, crisis-region and default-region just as before. We can also show that the no-default region in this case remains the same: before:

$$\boldsymbol{ND} = \left\{ \boldsymbol{b} \in \mathcal{B} \middle| \sum_n b_n \leq \bar{B} \right\}$$

where $\overline{B}$ is defined as in Proposition 2. Similarly, prices are one in this region, that is $q_n(V^D, \boldsymbol{b}) = 1$ for all n and $\boldsymbol{b} \in \boldsymbol{ND}$; and the value function is $V(\boldsymbol{b}, V^D) = u(y - r \sum_n b_n)/(1 - \beta)$ for all $\boldsymbol{b} \in \boldsymbol{ND}$.

With this in hand, we can now slightly redefine problem (PW) to account for the more general portfolio of maturities as follows:

$$\begin{aligned} W(\boldsymbol{b}, T) = \max_{\{b_{0,T} \in \mathbb{R}, \{c_t\}_{t=0}^{T-1}\}} & \left\{ \sum_{t=0}^{T-1} \beta^t (1-\lambda)^t u(c_t) \right. \\ & + \beta^T (1-\lambda)^{T-1} \left[\frac{u\left(y - r\left(b_{0,T} + \sum_{n \in \{1,\ldots,N\}} (1-\delta_n)^T b_n\right)\right)}{1-\beta} \right] \\ & \left. + \sum_{t=1}^{T-1} \beta^t (1-\lambda)^{t-1} \lambda \overline{V}^D \right\}, \end{aligned}$$

subject to:

$$\begin{aligned} b_0 \leq (1+r)^{-1} & \left[\sum_{t=0}^{T-1} \left(\frac{1-\lambda}{1+r} \right)^t \left(y - c_t - \sum_{n \in \{1,\ldots,N\}} (r + \delta_n)(1-\delta_n)^t b_n \right) \right. \\ & \left. + \left(\frac{1-\lambda}{1+r} \right)^{T-1} b_{0,T} \right], \\ b_{0,T} \leq \overline{B} - & \sum_{n \in \{1,\ldots,N\}} (1-\delta_n)^T b_n. \end{aligned}$$

and $c_t \geq 0$ for all t. As in the benchmark, the solution to this problem will feature a constant consumption, and where the last two constraints hold with equality.

Note that solution for problem W imposes the same requirement as before: the sovereign does not issue new long-term bonds of any $\delta_n < 1$, nor does it buy them back. For n such that $0 < \delta_n < 1$, the sovereign simply lets the long-term bonds mature, and for a perpetuity $\delta = 1$, there is no change in the amount outstanding. We now state the extension of our benchmark result to general portfolios:

Proposition 6. *In any Markov Equilibrium with a general portfolio, for any* $\boldsymbol{b} \in \boldsymbol{C}$*, we have that* $V(\boldsymbol{b}, \underline{V}^D) = \sup_{T \geq 1} W(\boldsymbol{b}, T)$ *as long as a solution to* $\sup_{T \geq 1} W(\boldsymbol{b}, T)$ *features* $b_{0,T} \geq 0$.

The above Proposition says that versions of Lemmas 1 and 2 hold for the general maturity case: that is, in any equilibrium, it is without loss of generality to restrict attention to sovereign strategies that use only the one-period bond, and do not actively participate in the long-term bond market. The last statement of the proposition just guarantees that the optimal solution to the W is possible; that is, features non-negative debt positions.

The above also highlights an important point: our result is about issuances, that is, during the periods of deleveraging, the country will not issue new long-term bonds. This is consistent with the fact that countries tend to switch their issuances towards the short-term end during periods where the interest rate spreads over risk free debt are high. However, whether or not the maturity of the overall portfolio of outstanding debt shortens or not during deleveraging depends on whether the pace at which long-term bonds come due is greater or less than the pace of deleveraging.

7 Conclusion

In this paper we have shown that actively engaging in the long-term bond market during periods of deleveraging entails costs for a sovereign. In particular, we have shown that shifts in the maturity structure may affect the incentives to develelerage and hence imply changes in the equilibrium prices of long-term bonds. Such changes are always moving against the borrower; that is, the price of the long-term bond rises when the sovereign buys them, while it falls when the sovereign issues more. Quite generally, these actions will tend to shrink the budget set of the borrower, generating an incentive to use only short-term bonds during a period of deleveraging.

The model we have described however does not completely rule out the use of long-term bonds, as there is a local indeterminacy in the optimal portfolio allocation in equilibrium due to the discrete time environment. That is, generically, small changes in the portfolio do not generate changes in the discrete exit times or prices, and hence, are also optimal. However, it is important to highlight that this result is driven by the discrete time assumption. Although we have refrained from presenting this formally, in the continuous-time limit the iso-T regions collapse to lines, and the deleveraging portfolio is uniquely pinned-down. In particular, any policy that involves long-term bonds is strictly suboptimal. Note however, that away from the region of deleveraging, in the interior of the $T(\boldsymbol{b}) = \infty$ region, the portfolio remains locally indeterminate, as small variations in the portfolio do not change the optimality of never exiting.

An important caveat in the above analysis is our focus on outside option shocks, with and without rollover crises. The literature has primarily focused on income (or endowment) shocks as the main source of uninsurable risk.[27] We have made this choice to transparently highlight how the incentives to deleverage, and the corresponding budgetary implications, are sensitive to maturity choice along the transition. In particular, maturity choice is not used to *hedge* risk in our environment. A hedging motive would arise if shocks affected consumption *absent* default, and if these changes in consumption had a non-zero covariance with bond prices. Our environment abstracts from this hedging motive as shocks do not change equilibrium consumption in periods of no default. The bonds in our environment are therefore not useful to hedge the risks we consider. In practice there are a richer set of assets to hedge risk beyond non-contingent bonds, and we do not want the absence of these assets influencing the core results. Finally, the desire to hedge is also operative in models of full commitment under incomplete markets, while our results arise exclusively due to limited commitment involving the speed of deleveraging and repayment. It is questionable how effectively the cyclical shifts in the slope of the yield curve can be exploited to insure the risks facing economies on the brink of default. For example, in the context of full commitment, Buera and Nicolini (2004) found the positions required to hedge are implausibly large (see also Faraglia et al., 2010, for a more recent analysis showcasing several problems with this approach). That said, we acknowledge that the desire to hedge may influence maturity choice, particularly in relatively tranquil periods in which the normal business cycles is the primary source of risk. We view our results as isolating an alternative force that also operates through equilibrium prices, but involves how maturity choice affects the incentives to deleverage and the corresponding equilibrium prices.

[27]There are important exceptions, see for example Cooley et al. (2004) and the more recent work of Hopenhayn and Werning (2008). For a more detailed summary of the literature, see the discussion in Aguiar and Amador (2013).

A Appendix: Proofs

A.1 Auxiliary Results

Many of the results exploit the fact that the value function conditional on repayment in the current period is continuous and monotonic in debt. Using standard arguments, we establish this result in part (a) of the following lemma. Part (b) of the lemma is useful in proving Proposition 2, which states that the no-default region is an absorbing region. The lemma uses the lenders' break-even conditions to establish an upper bound on the government's value for debt positions in this region. Moreover, the lemma establishes that *any* equilibrium strategy starting from this region that leads to default with positive probability generates a strictly lower utility. The proof uses the fact that the break-even conditions require that equilibrium prices compensate lenders for the possibility of default, as well as that endogenous default implies that $V(\boldsymbol{b}, V^D) < V^D$. The strict inequality statement in part (b) omits the case of $(b_S, b_L, V^D) = (0, \overline{B}, \overline{V}^D)$, which is easier to handle directly in the proof of Proposition 2.

Lemma A.1 (Continuity, Monotonicity and an Upperbound)**.** *In any Markov Perfect Equilibrium, for any $V^D \in \{\underline{V}^D, \overline{V}^D\}$ we have:*

(a) *For a given V^D, the function $(b_S, b_L) \mapsto V(b_S, b_L, V^D)$ is strictly decreasing and continuous in each argument; and*

(b) *Define $\overline{B}$ by $u(y - r\overline{B}) = (1-\beta)\overline{V}^D$. For any (b_S, b_L) such that $b_S + b_L \leq \overline{B}$, we have*

$$V(b_S, b_L, V^D) \leq \frac{u(y - r(b_S + b_L))}{1-\beta}.$$

This last inequality is strict if there is a default along some equilibrium path starting from (b_S, b_L, V^D) and any of the following three conditions hold: (i) $b_S > 0$, (ii) $b_S + b_L < \overline{B}$, or (iii) default occurs at $\underline{V}^D$.

Proof. **Part (a):** For monotonicity, consider a point b_S^0, b_L^0, and let $\epsilon_S \geq 0, \epsilon_L \geq 0$ with one and only one inequality strict. Then it follows that

$$\begin{aligned}
&V(b_S^0 - \epsilon_S, b_L^0 - \epsilon_L, V^D) \\
&\qquad \geq u(y - (1+r)b_S^0 - rb_L^0 + q_S b_S' + q_L(b_L' - b_L^0) + (1+r)\epsilon_S + r\epsilon_L) + \beta\mathbb{E}V \\
&\qquad > u(y - (1+r)b_S^0 - rb_L^0 + q_S b_S' + q_L(b_L' - b_L^0)) + \beta\mathbb{E}V \\
&\qquad = V(b_S^0, b_L^0, V^D),
\end{aligned}$$

where b_S' and b_L' represent the equilibrium debt policy when the state is (b_S^0, b_L^0, V^D); q_S and

q_L, represent the associated equilibrium price schedules evaluated at the debt policy choices; and $\mathbb{E}V$ is short-hand for the expected continuation value conditional on V^D and (b'_S, b'_L). The first line uses the fact that choices made for (b^0_S, b^0_L) are feasible (but not necessarily optimal) for $(b^0_S + \epsilon_S, b^0_L + \epsilon)$. The feasibility uses the fact that prices depend on (b'_S, b'_L, V^D) but not on b^0_j, $j = S, L$, so the choices (b'_S, b'_L) have the same budgetary consequences from either initial debt state. The second line uses the fact that $\epsilon_j \geq 0$, $j = S, L$, with one strictly greater than zero. The final line uses the fact that (b'_S, b'_L) represent the optimal choices at (b^0_S, b^0_L, V^D).

Continuity follows from a similar argument. Let $\boldsymbol{b}^A$ and $\boldsymbol{b}^B$ be two points in the state space. Let $\boldsymbol{b}^{A\prime}$ and $\boldsymbol{b}^{B\prime}$ be their respective optimal debt policies. Let $\boldsymbol{q}^A$ and $\boldsymbol{q}^B$ be the equilibrium vector of prices at which the optimal debt policies are traded. Let $\tilde{C}(\boldsymbol{b}, \boldsymbol{b}', \boldsymbol{q}) \equiv y - (1+r)b_S - rb_L + q_S b'_S + q_L(b'_L - b_L)$ denote the consumption implied by the initial debt $\boldsymbol{b}$ and the new debt policy $\boldsymbol{b}'$ issued at prices $\boldsymbol{q}$. Note that $|\tilde{C}(\boldsymbol{b}^A, \boldsymbol{b}', \boldsymbol{q}) - \tilde{C}(\boldsymbol{b}^B, \boldsymbol{b}', \boldsymbol{q})| \leq (1+r)|b^A_S - b^B_S| + (r + q_L)|b^A_L - b^B_L| \leq (1+r)\left(|b^A_S - b^B_S| + |b^A_L - b^B_L|\right)$. Then optimality implies:

$$V(\boldsymbol{b}^A, V^D) = u(\tilde{C}(\boldsymbol{b}^A, \boldsymbol{b}^{A\prime}, \boldsymbol{q}^A)) + \beta V(\boldsymbol{b}^{A\prime}, V^D) \geq u(\tilde{C}(\boldsymbol{b}^A, \boldsymbol{b}^{B\prime}, \boldsymbol{q}^B)) + \beta V(\boldsymbol{b}^{B\prime}, V^D).$$

Similarly:

$$V(\boldsymbol{b}^B, V^D) = u(\tilde{C}(\boldsymbol{b}^B, \boldsymbol{b}^{B\prime}, \boldsymbol{q}^B)) + \beta V(\boldsymbol{b}^{B\prime}, V^D) \geq u(\tilde{C}(\boldsymbol{b}^B, \boldsymbol{b}^{A\prime}, \boldsymbol{q}^A)) + \beta V(\boldsymbol{b}^{A\prime}, V^D).$$

Taking differences we have that:

$$\begin{aligned} & u(\tilde{C}(\boldsymbol{b}^A, \boldsymbol{b}^{A\prime}, \boldsymbol{q}^A)) - u(\tilde{C}(\boldsymbol{b}^B, \boldsymbol{b}^{A\prime}, \boldsymbol{q}^A)) \geq \\ & \qquad V(\boldsymbol{b}^A, V^D) - V(\boldsymbol{b}^B, V^D) \geq u(\tilde{C}(\boldsymbol{b}^A, \boldsymbol{b}^{B\prime}, \boldsymbol{q}^B)) - u(\tilde{C}(\boldsymbol{b}^B, \boldsymbol{b}^{B\prime}, \boldsymbol{q}^B)). \end{aligned}$$

Concavity implies that $|u(c') - u(c)| \leq \kappa |c' - c|$ where $\kappa = u'(\min(c, c'))$. This implies that as the distance between $\boldsymbol{b}^A$ and $\boldsymbol{b}^B$ goes to zero, so do the bounds above, implying the continuity of the value function.

Part (b): Starting from a $\boldsymbol{b} = (b_S, b_L)$ that satisfies the hypothesis of part (b), we can iterate forward on the equilibrium policy functions conditional on the path of V^D. In particular, denote histories of V^D realizations by $h^t \equiv (V^D_0, V^D_1, ..., V^D_t)$, with V^D_0 representing the current period realization. Let $\pi(h^t)$ denote the probability h^t is realized. Let $c(h^t)$ and $d(h^t)$ represent equilibrium consumption and default decisions conditional on h^t, where $d(h^t)$ equals one if the sovereign defaults at history h^t (but not before) and zero otherwise. It is convenient to define the "survival" indicator: $s(h^t) \equiv \prod_{h^s \in h^t}(1 - d(h^s))$, which equals one if there is no default along the path up to and including h^t and zero otherwise. Starting from $\boldsymbol{b}$, let (b'_S, b'_L) denotev the end-of-period equilibrium debt portfolio with associatedv equilibrium prices (q'_S, q'_L).

Equilibrium pricing for new debt requires:

$$q'_L = \sum_{t=1}^{\infty} R^{-t} \sum_{h^t} \pi(h^t) s(h^t) r, \tag{16a}$$

$$q'_S = \sum_{h^1} \pi(h^1) s(h^1). \tag{16b}$$

The value function can be written as:

$$V(\boldsymbol{b}, V^D) = u(c(h^0)) + \sum_{t=1}^{\infty} \beta^t \sum_{h^t} \pi(h^t) \left(s(h^t) u(c(h^t)) + d(h^t) V^D(h^t) \right). \tag{17}$$

The budget constraint, after using the equilibrium pricing at each history as well as the no-ponzi condition, is:

$$y - r b_L - c(h^0) + \sum_{t=1}^{\infty} R^{-t} \sum_{h^t} \pi(h^t) s(h^t)(y - r b_L - c(h^t)) = R b_S. \tag{18}$$

Equation (18) is a necessary condition in any equilibrium, and reflects the fact that new bondholders break even in expectation. This condition allows us to compute an upper bound for the value in (17). In particular, maximizing the right hand side of (17) over consumption processes, subject to (18), one obtains an upper bound to the value function given by:

$$V(\boldsymbol{b}, V^D) \leq u(c^\star) + \sum_{t=1}^{\infty} \beta^t \sum_{h^t} \pi(h^t)(s(h^t) u(c^\star) + d(h^t) V^D(h^t)) \tag{19}$$

where $c^\star = y - \frac{r}{r+q'_L} R b_S - r b_L$. Given that $c^\star \leq y - r(b_S + b_L)$ as $q'_L \leq 1$ and that $V^D(h^t) \leq u(y - r(b_S + b_L))/(1-\beta)$ (for $b_S + b_L \leq \overline{B}$), it follows then that $V(\boldsymbol{b}, V^D) \leq u(y - r(b_S + b_L))/(1-\beta)$, generating the upperbound.

Note that if there exists an h^t such that $d(h^t) = 1$, then $q'_L < 1$, and $c^\star < y - r(b_S + b_L)$ if $b_S > 0$, implying that $V(\boldsymbol{b}, V^D) < u(y - r(b_S + b_L))/(1-\beta)$. Note that if $b_S + b_L < \overline{B}$, or if $V^D(h^t) = \underline{V}^D$, the upperbound is also strict. $\square$

Proof of Proposition 1

The following is the proof of Proposition 1, where in addition, we show that, in any Markov equilibrium, $\boldsymbol{ND} = \{\boldsymbol{b} \in \mathcal{B} | b_S + b_L \leq \overline{B}\}$:

Proof. First note that the sets are disjoint by definition. Note that $\boldsymbol{ND}$ is non-empty as $V(0, 0, V^D) \geq u(y)/(1-\beta) > V^D$ for any V^D. Now, Lemma A.1 implies that $\boldsymbol{ND} \subset \{\boldsymbol{b} \in \mathcal{B} | b_S + b_L \leq \overline{B}\}$, as strict monotonicity implies that $V(b_S, b_L, \overline{V}^D) < \overline{V}^D$ for $b_S + b_L > \overline{B}$. Now, note that for any $\boldsymbol{b} \in \boldsymbol{ND}$, $q_s(b) = 1$, which implies that $V(\boldsymbol{b}, V^D) \geq u(y - r(b_S + b_L))/(1-\beta)$, as the sovereign could choose to roll over its short-term debt at a risk free price while remaining in $\boldsymbol{ND}$ forever. Hence the bound of Lemma A.1 part (b) binds, and $V(\boldsymbol{b}, V^D) = u(y - r(b_S + b_L))/(1-\beta)$ for all $\boldsymbol{b} \in \boldsymbol{ND}$ and all V^D. Continuity and strict monotonicity of the no-default value function (as established by Lemma A.1 part a), implies that the boundary of the $\boldsymbol{ND}$ region is such that $V(\boldsymbol{b}, V^D) = \overline{V}^D$, which delivers $\boldsymbol{ND} = \{\boldsymbol{b} \in \mathcal{B} | b_S + b_L \leq \overline{B}\}$. On the complement of $\boldsymbol{ND}$, strict monotonicity of V delivers that $V(\boldsymbol{b}, V^D) < \overline{V}^D$ for any V^D, hence the complement of $\boldsymbol{ND}$ can be partitioned into $\boldsymbol{C}$ and $\boldsymbol{D}$. Given that $\underline{V}^D < \overline{V}^D$, by continuity, there exists a non-empty $\boldsymbol{C}$ region. Finally, the assumption that $u(0) < (1-\beta)\underline{V}^D$ implies that for sufficiently high $\boldsymbol{b}$, the no-default value will necessarily lie below $\underline{V}^D$, establishing a non-empty $\boldsymbol{D}$. □

A.2 Proof of Propositions 2

We first consider the absorbing state $V^D = \overline{V}^D$, and prove the following Lemma.

Lemma A.2. *In any Markov Perfect Equilibrium, for* $(b_S, b_L) \in \boldsymbol{ND}$ *we have*

$$V(b_S, b_L, \overline{V}^D) = \frac{u(y - r(b_S + b_L))}{1-\beta};$$

$$q_S(b_S, b_L, \overline{V}^D) = q_L(b_S, b_L, \overline{V}^D) = \begin{cases} 0 & \textit{if } (b_S, b_L) \notin \boldsymbol{ND} \\ 1 & \textit{if } (b_S, b_L) \in \boldsymbol{ND}; \end{cases}$$

$$D(b_S, b_L, \overline{V}^D) = \begin{cases} 1 & \textit{if } (b_S, b_L) \notin \boldsymbol{ND} \\ 0 & \textit{if } (b_S, b_L) \in \boldsymbol{ND}; \end{cases}$$

$$B_S(b_S, b_L, \overline{V}^D) + B_L(b_S, b_L, \overline{V}^D) = b_S + b_L,$$

Proof. The first equality was established in the proof of Proposition 1. The definition of the regions, imply the default policy above, as well as the prices of the short-term bond. The default policy also implies that the price of the long-term bond, $q_L = 0$ for $\boldsymbol{b} \not{\!\!\!/} in \boldsymbol{ND}$. Our tie-breaking assumption in footnote 8 states that the government does not issue bonds at zero prices, which implies that bond policy functions on $\boldsymbol{b} \in \boldsymbol{ND}$ satisfy $B_S(\boldsymbol{b}, \overline{V}^D) + B_L(\boldsymbol{b}, \overline{V}^D) \leq \overline{B}$. This implies that $\boldsymbol{ND}$ is an absorbing state, and thus $q_L = 1$ on $\boldsymbol{ND}$. Given the above, the only way to achieve the upper bound in $\boldsymbol{ND}$ is to maintain a stationary level of total debt $b_S + b_L$. To see this, note that:

$$V(b_S, b_L, \overline{V}^D) = \max_{(b'_S, b'_L) \in \boldsymbol{ND}} u(y - r(b_S - b_L) + b'_S - b_S + b'_L - b_L) + \beta \frac{u(y - r(b'_S + b'_L))}{1 - \beta}.$$

By strict concavity of u, the maximum is only achieved when $b_S + b_L = b'_S + b'_L$, establishing the final claim in the lemma. $\square$

We now turn to the case of $V^D = \underline{V}^D$, and prove the following lemma:

Lemma A.3. *In any Markov Perfect Equilibria, for $(b_S, b_L) \in \boldsymbol{ND}$, we have that:*

$$\begin{aligned}
V(b_S, b_L, \underline{V}^D) &= \frac{u(y - r(b_S + b_L))}{1 - \beta}; \\
q_S(b_S, b_L, \underline{V}^D) &= q_L(b_S, b_L, \underline{V}^D) = 1; \\
D(b_S, b_L, \underline{V}^D) &= 0; \\
B_S(b_S, b_L, \underline{V}^D) + B_L(b_S, b_L, \underline{V}^D) &= b_S + b_L.
\end{aligned}$$

Proof. The first equality was established in the proof of Proposition 1. The definition of the regions, imply the default policy above, as well as the prices of the short-term bond.

We now show that default does not occur along the equilibrium path starting from $(\boldsymbol{b}, \underline{V}^D)$ for $\boldsymbol{b} \in \boldsymbol{ND}$. Part (b) of Lemma A.1 establishes that if either $b_S + b_L < \overline{B}$, or $b_S > 0$, or default occurs at $\underline{V}^D$, it follows that the upper bound cannot be achieved in equilibrium, contradicting the result that $V(\boldsymbol{b}, V^D)$ achieves this upper bound on $\boldsymbol{ND}$. For these cases, the no-default region is absorbing and $q_L(b_S, b_L, \underline{V}^D) = 1$.

This leaves one remaining case; namely, the case of $(b_S, b_L) = (0, \overline{B})$ and default occurs when $V^D = \overline{V}^D$. To show that default in this case is strictly dominated by the upper bound, suppose that the government defaults at the start of period $t + 1$ at some history $\{h^t, \overline{V}^D\}$ starting from the initial state $(b_S, b_L, V^D) = (0, \overline{B}, \underline{V}^D)$. We can state a number of facts about values and debt positions at the start of period $t + 1$. First, for default to be optimal at $\{h^t, \overline{V}^D\}$, the tie-breaking assumptions requires $V(\{h^t, \overline{V}^D\}) < \overline{V}^D$. By strict monotonicity of V, this implies that the debt position at the start of period $t + 1$ lies outside $\boldsymbol{ND}$. Moreover, we have that in the alternative strong-enforcement continuation from h^t, namely $\{h^t, \underline{V}^D\}$, default would not have occurred in period $t + 1$. This follows from the tie-breaking assumption that the sovereign will not borrow into a region where it will default with probability one, as this will never be strictly optimal (see footnote

8). That is, the history $\{h^t, \underline{V}^D\}$ occurs with strictly positive probability starting from the initial state and debt lies outside $\boldsymbol{ND}$ at this history. The fact that debt lies outside $\boldsymbol{ND}$ at the end of period t plus strict monotonicity implies that $V(\{h^t, \underline{V}^D) < \overline{V}^D$. By definition of $\overline{B}$, this last inequality also implies that $V(\{h^t, \underline{V}^D) < u\left(y - r\overline{B}\right)/(1-\beta)$. This implies that consumption is not always $y - r\overline{B}$ for every continuation history starting from $\{h^t, \underline{V}^D\}$. More formally, let $\{c(h^i)\}_{i=0}^{\infty}$ denote the sequence of consumption starting from $(b_S, b_L, V^D) = (0, \overline{B}, \underline{V}^D)$. Then there exists some history h^τ, $\tau \geq t+1$, following history $\{h^t, \underline{V}^D\}$, such that $c(h^\tau) \neq y - r\overline{B}$. Using strict concavity of u, together with (19), this implies that the value function $V(0, \overline{B}, \underline{V}^D) < u(y - r\overline{B})/(1-\beta)$. This implies that equilibrium default along any history starting from $(b_S, b_L, \underline{V}^D) = (0, \overline{B}, \underline{V}^D)$ yields a value strictly less than the upper bound. The fact that the upper bound is feasible implies that default will never occur in equilibrium starting from this initial state, and $q_j(0, \overline{B}, \underline{V}^D) = 1$, $j = S, L$, completing the characterization of the no-default region.

Given that $\boldsymbol{ND}$ is an absorbing state (as default cannot occur in the equilibrium path), we have that

$$V(b_S, b_L, \underline{V}^D) = \max_{(b'_S, b'_L) \in \boldsymbol{ND}} \left\{ u(y - r(b_S - b_L) + b'_S - b_S + b'_L - b_L) \right. \\ \left. + \beta \frac{u(y - r(b'_S + b'_L))}{1-\beta} \right\}.$$

The maximum is only achieved when $b_S + b_L = b'_S + b'_L$, establishing the final claim in the lemma. □

A.3 Proof of Lemma 1

Proof. Let's do the case where $T(\boldsymbol{b}) < \infty$ (the case where $T(\boldsymbol{b}) = \infty$ works in a similar fashion). Let $\{c_t\}_{t=0}^{T-1}$ denote the consumption stream associated with $V(\boldsymbol{b}, \underline{V})$, with associated sequence of debt positions $\{b_{S,t}, b_{L,t}\}_{t=0}^{T}$. By definition, $(b_{S,0}, b_{L,0}) = (b_S, b_L) = \boldsymbol{b}$ and define $B_T \equiv b_{S,T} + b_{L,T}$ as the bond position on entry to the safe zone. In the safe zone, maturity is irrelevant and all that matters is the total stock of debt. Thus B_T is a sufficient statistic for welfare at the start of $t = T$. For $t \leq T-1$, utility depends on the risk of default and the consumption stream absent default. Exit time T is sufficient to pin down the risk of default regardless of the detailed allocation. We therefore focus on the consumption allocation absent default while in the crisis zone.

Letting $q_{i,t+1} = q_i(\boldsymbol{b}_t, \underline{V}^D, \boldsymbol{b}_{t+1})$, $i = S, L$, the budget constraint for t is:

$$c_t = y - (1+r)b_{s,t} - rb_{L,t} + q_{S,t+1}b_{S,t+1} + q_{L,t+1}(b_{L,t+1} - b_{L,t}).$$

For $t < T-1$, (2) and (3) imply $q_{S,t+1} = 1 - \lambda$ and $r = \left(\frac{1+r}{1-\lambda}\right) q_{L,t} - q_{L,t+1}$. Substituting in, we have

$$c_t = y - \left(\frac{1+r}{1-\lambda}\right)(q_{S,t}b_{S,t} + q_{L,t}b_{L,t}) + (q_{S,t+1}b_{S,t+1} + q_{L,t+1}b_{L,t+1}), \; t < T-1.$$

This is a difference equation in the total market value of debt at the end of period t: $q_{S,t+1}b_{S,t+1} + q_{L,t+1}b_{L,t+1}$. Solving forward to $t = T-2$ starting from $t = 0$, we have:

$$\begin{aligned} q_{S,0}b_{S,0} + q_{L,0}b_{L,0} = &\left(\frac{1-\lambda}{1+r}\right)\sum_{t=0}^{T-2}\left(\frac{1-\lambda}{1+r}\right)^t (y - c_t) \\ &+ \left(\frac{1-\lambda}{1+r}\right)^{T-1}(q_{S,T-1}b_{S,T-1} + q_{L,T-1}b_{L,T-1}). \end{aligned}$$

The budget constraint at $t = T-1$ is $c_{T-1} = y - (1+r)(b_{S,T-1} + b_{L,T-1}) + B_T$, where $B_T = b_{S,T} + b_{L,T}$ is the market value of debt on entry into the no-default zone. Using this plus $q_{i,T-1} = 1 - \lambda, i = S, L$, to replace $q_{S,T-1}b_{S,T-1} + q_{L,T-1}b_{L,T-1}$ in the above, we have

$$q_{S,0}b_{S,0} + q_{L,0}b_{L,0} = \left(\frac{1-\lambda}{1+r}\right)\sum_{t=0}^{T-1}\left(\frac{1-\lambda}{1+r}\right)^t (y - c_t) + \left(\frac{1-\lambda}{1+r}\right)^T B_T.$$

From (2) we have $q_{S,0} = 1 - \lambda$. From (3) we have

$$q_{L,1} = r\sum_{t=1}^{T-1}\left(\frac{1-\lambda}{1+r}\right)^t + \left(\frac{1-\lambda}{1+r}\right)^{T-1},$$

where $q_{L,1}$ is the price of bonds issued at the end of period zero. Using $q_{L,0} = \left(\frac{1-\lambda}{1+r}\right)(r + q_{L,1})$, we have after re-arranging:

$$b_{S,0} = \left(\frac{1}{1+r}\right)\left(\sum_{t=0}^{T-1}\left(\frac{1-\lambda}{1+r}\right)^t (y - c_t - rb_{L,0}) + \left(\frac{1-\lambda}{1+r}\right)^{T-1}(B_T - b_{L,0})\right).$$

Thus the allocation $\{c_t\}$ satisfies the budget constraint for the W problem (PW). As W is the maximum over all such feasible consumption streams, the result follows. $\square$

A.4 Proof of Claim 1

Proof. Using that $\overline{V}^D = u(y - r\overline{B})/(1-\beta)$, together with equation (5), we have that:

$$\begin{aligned}(1-\beta(1-\lambda))(W(b,\infty)-W(b,T)) &= u(y - rb_L - (r+\lambda)b_S) \\ &\quad - \Bigg[(1-\beta^T(1-\lambda)^T)\times \\ u\left(y - rb_L - \frac{(1-\beta(1-\lambda))(b_S(1-\lambda)+(b_L-\overline{B})\beta^T(1-\lambda)^T)}{\beta(1-\lambda)(1-\beta^T(1-\lambda)^T)}\right) & \\ &\quad + \beta^T(1-\lambda)^T u(y - r\overline{B})\Bigg].\end{aligned}$$

Using concavity of u it follows that:

$$\begin{aligned}&(1-\beta(1-\lambda))(W(b,\infty)-W(b,T)) \geq u(y - rb_L - (r+\lambda)b_S) \\ &\quad - u\Bigg((1-\beta^T(1-\lambda)^T)\left[y - rb_L - \frac{(1-\beta(1-\lambda))(b_S(1-\lambda)+(b_L-\overline{B})\beta^T(1-\lambda)^T)}{\beta(1-\lambda)(1-\beta^T(1-\lambda)^T)}\right] \\ &\qquad + \beta^T(1-\lambda)^T\left[y - r\overline{B}\right]\Bigg) \\ &\quad = u(y - rb_L - (r+\lambda)b_S) - u\left(y - (r+\lambda)b_S - rb_L + (\overline{B}-b_L)\lambda\beta^{T-1}(1-\lambda)^{T-1}\right) \geq 0,\end{aligned}$$

where the last line follows from $b_L > \overline{B}$. So it follows then that $T=\infty$ is an optimal solution of Problem (PW) as long as $b_L > \overline{B}$ and $\boldsymbol{b} \in \boldsymbol{C}$. □

A.5 Proof of Lemma 2

Proof. Consider the case where $T < \infty$. Let $\{\hat{c}_t\}_{t=0}^{T-1}$ and $\hat{b}_{S,T}$ be the solution to Problem (PW) for a given T and $\boldsymbol{b} = (b_S, b_L) \in \boldsymbol{C}$. Let $\{\hat{b}_{S,t}\}_{t=1}^{T-1}$ be defined recursively as:

$$\hat{b}_{S,t} = \frac{1}{1-\lambda}\left(y - c_t - rb_L - R\hat{b}_{S,t-1}\right)$$

where $\hat{b}_{S,0} = b_S$. Note, as argued before, an optimal solution $(\hat{b}_{S,t}, b_L) \in \boldsymbol{C}$ for $t \leq T-1$.

The sovereign's problem in any equilibrium satisfies the following Bellman equation:

$$V((b_S,b_L),\underline{V}^D) = \max_{\{c,b'_S,b'_L\}} u(c) + \beta\left(1 - \lambda\mathbf{1}_{\{b'_S+b'_L>\overline{B}\}}\right)V(\boldsymbol{b}',\underline{V}^D) + \beta\lambda\mathbf{1}_{\{b'_S+b'_L>\overline{B}\}}\overline{V}^D,$$

subject to

$$c \leq y - (1+r)b_S - rb_L + q_S(\boldsymbol{b}')b'_S + q_L(\boldsymbol{b}')(b'_L - b_L). \tag{20}$$

Let $c = \hat{c}_0$, $b'_S = \hat{b}_{S,1}$ and $b'_L = b_L$. Those choices satisfy the budget constraint, as long as $(\hat{b}_{S,1}, b_L) \in \boldsymbol{C}$, given that $q_S(\hat{b}_{S,1}, b_L) = 1 - \lambda$, and it follows then that such choice provides a lower bound on the utility from the Bellman equation:

$$V((b_S, b_L), \underline{V}^D) \geq u(\hat{c}_0) + \beta(1-\lambda)V((\hat{b}_{S,1}, b_L), \underline{V}^D) + \beta\lambda\overline{V}^D$$

But now consider, the value function $V(\hat{b}_{S,1}, b_L, \underline{V}^D)$. It is the case that for that state, the choice of $\hat{c}_1$ and $b'_S = \hat{b}_{S,2}$ and $b'_L = b_L$ satisfies the budget constraint (as long as $(\hat{b}_{S,2}, b_L) \in \boldsymbol{C}$), and hence it follows that this provides a lower bound as well:

$$V((\hat{b}_{S,1}, b_L), \underline{V}^D) \geq u(\hat{c}_1) + \beta(1-\lambda)V((\hat{b}_{S,2}, b_L), \underline{V}^D) + \beta\lambda\overline{V}^D$$

Keeping iterating, we have that in the last period before exiting, $T-1$, the value function is bounded below by:

$$V((\hat{b}_{S,T-1}, b_L), \underline{V}^D) \geq u(\hat{c}_{T-1}) + \beta(1-\lambda)V((\hat{b}_{S,T}, b_L), \underline{V}^D) + \beta\lambda\overline{V}^D$$

Now, note that $V((\hat{b}_{S,T}, b_L), \underline{V}^D) = u(y - r(\hat{b}_{S,T} + b_L))/(1-\beta)$, given that $\hat{b}_{S,T} + b_L \leq \overline{B}$. It follows then that:

$$\begin{aligned} V((b_S, b_L), \underline{V}^D) \geq & \sum_{t=0}^{T-1} \beta^t (1-\lambda) u(\hat{c}_t) + \\ & + \beta^T (1-\lambda)^{T-1} \frac{u(y - r(\hat{b}_{S,T} + b_L))}{1-\beta} + \sum_{t=1}^{T-1} \beta^t (1-\lambda)^{t-1} \lambda \overline{V}^D = W(\boldsymbol{b}, T) \end{aligned}$$

by putting all the inequalities above together.

So we have that $V(\boldsymbol{b}, \underline{V}^D) \geq W(\boldsymbol{b}, T)$ for any finite $T \geq 1$. A similar argument, together with boundedness of V, shows that $V(\boldsymbol{b}, \underline{V}^D) \geq W(\boldsymbol{b}, \infty)$. And the result of the lemma follows. □

A.6 Proof of Theorem 1

Proof. The proof of Theorem 1 follows from Lemmas 1 and 2. □

A.7 Proof of Lemma 3

Proof. Consider a zero-cost trade from point $\boldsymbol{b}$ to $\boldsymbol{b}'$. Note that if $\boldsymbol{b} \in \boldsymbol{C}$, then $\boldsymbol{b}' \in \boldsymbol{C}$, as the sovereign cannot enter the no-default zone through a zero-cost trade at risk-free prices. Thus $q_S(A) = q_S(B) = 1 - \lambda$. Define $q_L(T)$ for a given T from equation (3) and substitute into equation (4) to obtain:

$$C^T = -\left(\frac{1-\beta(1-\lambda)}{\beta(1-\lambda)(1-\beta^T(1-\lambda)^T)}\right)[q_L(T)b_L' + q_S(T)b_S'] + y - \left(\frac{1-\beta(1-\lambda)}{1-\beta^T(1-\lambda)^T}\right)\overline{B}.$$

Note that a zero-cost trade implies $q_L(\boldsymbol{b}')b_L' + q_S(\boldsymbol{b}')b_S' = q_L(\boldsymbol{b}')b_L + q_S(\boldsymbol{b}')b_S$, thus we can substitute $\boldsymbol{b}$ for $\boldsymbol{b}'$ in the above bracketed expression if $T = T(\boldsymbol{b}')$. This implies that it is feasible for the sovereign to exit in $T(\boldsymbol{b}')$ periods from $\boldsymbol{b}$ without changing consumption. Thus $W(\boldsymbol{b}, T(\boldsymbol{b}')) = W(\boldsymbol{b}', T(\boldsymbol{b}')) = V(\boldsymbol{b}')$. However, $V(\boldsymbol{b}) = \max_{T'} W(\boldsymbol{b}, T') \geq W(\boldsymbol{b}, T(\boldsymbol{b}'))$, with strict inequality if the sovereign has a unique optimal $T(\boldsymbol{b}) \neq T(\boldsymbol{b}')$. □

A.8 Proof of Proposition 3

Proof. Lemma A.2 obtains the value function for $\boldsymbol{b} \in \boldsymbol{ND}$ region, when $V^D = \overline{V}^D$. For the rest of the domain in this case, we can use the strict monotocity obtained in Lemma A.1 to argue that the value function must be below $\overline{V}^D$ for $\boldsymbol{b} \notin \boldsymbol{ND}$; given that at the boundary of $\boldsymbol{ND}$, the value function equals $\overline{V}^D$.

Lemma A.3 obtains the value function for $\boldsymbol{b} \in \boldsymbol{ND}$ region, when $V^D = \underline{V}^D$. Using Proposition 1 allows to characterized the value function as equal to $\sup_T W(\boldsymbol{b}, T)$ for $\boldsymbol{b} \in \boldsymbol{C}$ and when $V^D = \underline{V}^D$. Finally, exploiting the strict monotonicity of the value function, as given by Lemma A.1, it follows that the value function is strictly less than $\underline{V}$ for $\boldsymbol{b} \in \boldsymbol{D}$ and $V^D = \underline{V}^D$; which completes the proof of the Proposition. □

A.9 Proof of Propositions 4 and 5

Proof. The proof for prices and values when $V^D = \overline{V}^D$ is done in the main body of the text. Let us now argue that the conjectured prices and value functions are an equilibrium as well when $V^D = \underline{V}^D$. To do this, note that the equilibrium value of reaching $b_S + b_L = \bar{B}$ equals $u(y - r\bar{B})/(1-\beta)$. This follows because, as explained in the body of the text, for any $b_S' + b_L' = \bar{B}$, there exist a zero cost trade that moves the country debt position to $(0, \bar{B})$, guaranteeing the maximum possible payoff in that region, $u(y - r\bar{B})/(1-\beta)$. Hence, the arguments of Lemma 1 and 2 hold and we can pinned down the value function in the $\boldsymbol{C}$ region by $\sup_T W(\bar{b}, T)$, just as before. Note that the same way that we defined $\boldsymbol{D}$ in the previous equilibrium is consistent with an equilibrium in this environment as well.

It remains to show that that when $\boldsymbol{b} \in \boldsymbol{ND} - \tilde{\boldsymbol{ND}}$, the equilibrium value is the same as in the previous case. To see this, just note that for such a $\boldsymbol{b}$, there exists a zero cost trade that at prices of 1 moves the debt position to the $\tilde{\boldsymbol{ND}}$ region, guaranteeing a payoff of $u(y - r(b_S + b_L))/(1 - \beta)$, which is again, the maximum possible value in equilibrium in that region.

Finally, the price of the long term bond needs to be adjusted for the one period risk that country has of defaulting when $\boldsymbol{b} \in \boldsymbol{ND} - \tilde{\boldsymbol{ND}}$, which only lasts for one period, as the country will immediately trade to the no default region $\tilde{\boldsymbol{ND}}$. □

A.10 Proof of Proposition 6

Proof. Equilibrium prices for a bond with "maturity" δ_n satisfies the difference equation (where $q_n = q_n(\underline{V}^D, \boldsymbol{b}'))$:

$$q_n = (1 - \lambda)\beta(r + \delta_n + (1 - \delta_n)q_n'),$$

if $\boldsymbol{b}' \in \boldsymbol{C}$, and $q_n = 1$ if $\boldsymbol{b}' \in \boldsymbol{ND}$. Solving forward from $t = 0$ to $t = T \geq 1$, we have:

$$q_{n,0} = \beta(1 - \lambda)(r + \delta_n) \sum_{t=0}^{T-1} \beta^t (1 - \lambda)^t (1 - \delta_n)^t + \beta^T (1 - \lambda)^T (1 - \delta_n)^T q_{n,T}. \tag{21}$$

If T is the exit time, then $q_{n,T} = 1$.

The budget constraint for the equilibrium problem (where $T = T(\boldsymbol{b})$) satisfies (for $t \leq T - 1$):

$$\begin{aligned} c_t &= y - \sum_{n=0}^{N} (r + \delta_n) b_{n,t} + \sum_{n-0}^{N} q_{n,t+1} \left(b_{n,t+1} - (1 - \delta_n) b_{n,t}\right) \\ &= y - \sum_{n=0}^{N} \underbrace{(r + \delta_n + (1 - \delta_n) q_{n,t+1})}_{\frac{q_{n,t}}{\beta(1-\lambda)}} b_{n,t} + \sum_{n=0}^{N} q_{n,t+1} b_{n,t+1}. \end{aligned}$$

Rearranging:

$$\sum_{n=0}^{N} q_{n,t} b_{n,t} = \beta(1 - \lambda)(y - c_t) + \beta(1 - \lambda) \sum_{n=0}^{N} q_{n,t+1} b_{n,t+1}$$

Define $MV_t \equiv \sum_{n=0}^{N} q_{n,t} b_{n,t}$ to be the market value of debt at the end of period $t - 1$. The above expression becomes

$$MV_t = \beta(1-\lambda)(y-c_t) + \beta(1-\lambda)MV_{t+1}.$$

Solve forward from $t=0$ to $t=T-1$:

$$MV_0 = \beta(1-\lambda)\sum_{t=0}^{T-1}\beta^t(1-\lambda)^t(y-c_t) + \beta^T(1-\lambda)^T MV_T. \tag{22}$$

Using the pricing equation (21), we also have:

$$\begin{aligned}
MV_0 &= \sum_{n=0}^{N} q_{n,0}b_{n,0} \\
&= (1-\lambda)b_{0,0} + \\
&\sum_{n=1}^{N} b_{n,0}\left(\beta(1-\lambda)(r+\delta_n)\sum_{t=0}^{T-1}\beta^t(1-\lambda)^t(1-\delta_n)^t + \beta^T(1-\lambda)^T(1-\delta_n)^T\right).
\end{aligned}$$

Substituting into (22) and rearranging, we have:

$$\begin{aligned}
b_{0,0} = \beta\Bigg[&\sum_{t=0}^{T-1}\beta^t(1-\lambda)^t\left(y-c_t-\sum_{n=1}^{N}b_{n,0}\left(r+\delta_n\right)\left(1-\delta_n\right)^t\right) \\
&+\beta^{T-1}(1-\lambda)^{T-1}\left(MV_T - \sum_{n=1}^{N}b_{n,0}(1-\delta_n)^T\right)\Bigg].
\end{aligned}$$

Note that by definition of $T = T(\boldsymbol{b})$, we have $MV_T = \sum_{n=0}^{N} b_{n,T} \leq \bar{B}$. Thus, equilibrium consumption satisfies the budget constraint for the $W(\boldsymbol{b}, T(\boldsymbol{b}))$ problem with $b_{0,T} = MV_T - \sum_{n=1}^{N} b_{n,0}(1-\delta_n)^T \leq \bar{B} - \sum_{n=1}^{N} b_{n,0}(1-\delta_n)^T$. Thus, $W(\boldsymbol{b}, T(\boldsymbol{b})) \geq V(\boldsymbol{b}, \underline{V}^D)$ for $\boldsymbol{b} \in \boldsymbol{C}$.

Going the other way, any consumption sequence and $b_{0,T}$ that satisfies the constraints of the $W(\boldsymbol{b}, T)$ problem for any T that is feasible also satisfies the sequence of budget constraints for the V problem. Thus the $W(\boldsymbol{b}, T)$ value is feasible in equilibrium, and so $V(\boldsymbol{b}, \underline{V}^D) \geq W(\boldsymbol{b}, T)$ for all T. □

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